FINITE ELEMENT METHODS

CHAPTER 2 – BARS AND BEAMS

LINEAR STATIC ANALYSIS [1]

Course: Eindige Element Metodes (EEM) 414, Department of Mechanical Engineering, Stellenbosch University
Textbook: Finite Element Modelling for Stress Analysis, 1995, Robert D. Cook, John Wiley & Sons, Inc.
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[1] Finite Element Modelling for Stress Analysis, 1995, Robert D. Cook, John Wiley & Sons, Inc.
Chapter and section numbering follows that of Cook, 1995. Additional information is given using numbers starting with "B".

The textbook starts immediately with the description of bar and beam elements. We will first do an even simpler element – the *spring* element. The reason for this is that we already know the stiffness of the spring k. With several springs in a system, the global stiffness matrix needs to be *assembled*. This procedure can easily be explained using spring elements. We will then move onto the bar and beam elements where we will calculate the stiffness matrix for each respectively. The process of assembling the element stiffness matrices into a global stiffness matrix is the same for any type of element.

B.1 Spring Elements and Assembly

B.1.1 Single Element

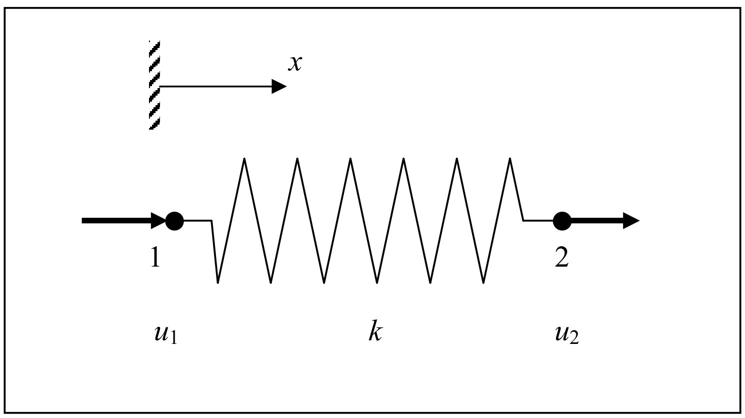


Figure B.1-1 – Single spring element with stiffness k

Assume that the spring is linear elastic, following the relation $F = k\Delta$. Let node 2 be displaced by a distance u_2 and node 1 by a distance u_1 with $u_2 > u_1$. Consider the equilibrium of forces for the spring under these displacements.

$$f_1 = -k(u_2 - u_1)$$

$$f_2 = k(u_2 - u_1) \qquad \text{B.1-1}$$

Equation B.1-1 can be written in matrix notation,

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad \text{B.1-2}$$

stiffness matrix x *displacement vector* = *force vector*

Lets analyse Equation B.1-2 by setting $u_2 = 0$,

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} u_1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} k \\ -k \end{bmatrix} u_1 \qquad \text{B.1-3a}$$

and similarly by setting $u_1 = 0$,

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix} u_2 \qquad \text{B.1-3b}$$

From this it can be seen that the first column in the stiffness matrix gives the force at each node in order to have a unit displacement at node 1 and zero displacement at all the other nodes. Similarly, the second column in the stiffness matrix gives the nodal forces with a unit displacement at node 2 and zero displacement at all the other nodes. This procedure will be used later to determine the stiffness matrix for beam elements.

B.1.2 System of Spring Elements

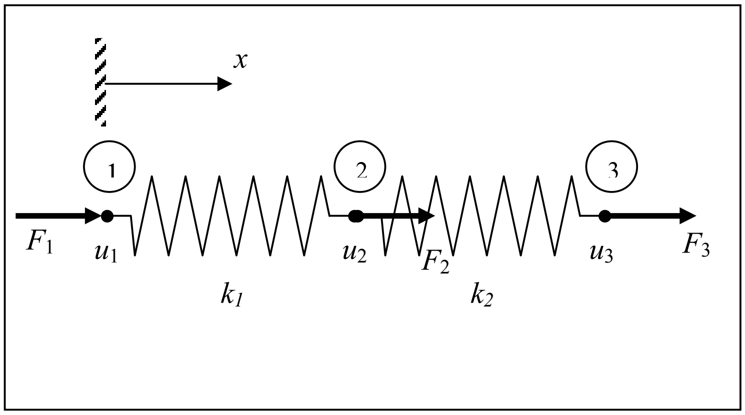


Figure B.1-2 – System of single spring elements

For element 1, the element nodes are 1 and 2 as in Figure B.1-1. These nodes correspond to the global nodes 1 and 2, as indicated in Figure B.1-2. For this element, the equilibrium equation is given by

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1^{ele1} \\ f_2^{ele1} \end{bmatrix} \qquad \text{B.1-4a}$$

For element 2, the element nodes 1 and 2 correspond to global nodes 2 and 3 respectively,

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_2^{ele2} \\ f_3^{ele2} \end{bmatrix} \qquad \text{B.1-4b}$$

Equilibrium of nodal forces can be written as

$$F_1 = f_1^{ele1}, \qquad F_2 = f_2^{ele1} + f_2^{ele2}, \qquad F_3 = f_3^{ele2} \qquad \text{B.1-5}$$

where F indicates global nodal forces and f element nodal forces. Making use of Equation B.1-4, the equilibrium equations Equation B.1-5 follows,

$$
\begin{aligned}
F_1 &= f_1^{ele1} &&= k_1 u_1 - k_1 u_2, \\
F_2 &= f_2^{ele1} + f_2^{ele2} &&= -k_1 u_1 + (k_1 + k_2) u_2 - k_2 u_3, \\
F_3 &= f_3^{ele2} &&= -k_2 u_2 + k_2 u_3
\end{aligned}
\qquad \text{B.1-6}
$$

In matrix form

$$
\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
=
\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}
\qquad \text{B.1-7}
$$

$$
\underline{\overline{K}}\,\overline{D} = \overline{F}
$$

The two individual stiffness matrices for element 1 and element 2 are indicated in Equation B.1-7. Equation B.1-7 is the global set of equations, with $\underline{\overline{K}}$ the global stiffness matrix, $\overline{D}$ the global nodal displacements and $\overline{F}$ the global nodal force vector.

An alternative way of assembling the global stiffness matrix follows: The stiffness matrix for each element can be "enlarged",

$$
\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
=
\begin{bmatrix} f_1^{ele1} \\ f_2^{ele1} \\ 0 \end{bmatrix}
\quad \text{for element 1}
$$

$$
\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
=
\begin{bmatrix} 0 \\ f_2^{ele2} \\ f_3^{ele2} \end{bmatrix}
\quad \text{for element 2}
\qquad \text{B.1-8}
$$

The two equations can now simply be added together,

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1^{ele1} \\ f_2^{ele1} + f_2^{ele2} \\ f_3^{ele2} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \qquad \text{B.1-9}$$

B.1.3 Boundary Conditions

Can Equation B.1-9 be solved for the nodal displacements? In order to accomplish this, we need to calculate the inverse of the stiffness matrix, $\overline{D} = \underline{\overline{K}}^{-1}\overline{R}$

The determinant of the stiffness matrix is given by,

$$\det \underline{\overline{K}} \quad = \quad k_1\left((k_1+k_2)k_2 - k_2^2\right) - k_1^2 k_2 \quad = \quad k_1^2 k_2 + k_2^2 - k_2^2 - k_1^2 k_2 = 0 \qquad \text{B.1-10}$$

The determinant is equal to zero, which means that the inverse of $\underline{\overline{K}}$ is not defined and the nodal displacements can not be solved for.

In physical terms this means that the system (springs) are not constrained, i.e. rigid body motion is possible. To avoid rigid body motion, constraints should be applied. The most common constraint is to “fix” one or more of the nodes.

For this example, let’s fix node 1: $u_1 = 0$ and $F_2 = F_3 = P$. The external force applied at node 1 would be the reaction force, an unknown. Equation B.1-7 can be written as

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ P \\ P \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} P \\ P \end{bmatrix} \qquad \text{B.1-11}$$

and $F_1 = -k_1 u_2$, the reaction force. This is equivalent to deleting row 1 and column 1 in the stiffness matrix. If it was node 2 that was constrained, it would have been row 2 and column 2 that would have been deleted.

The *reduced* stiffness matrix is no longer singular, and the system can be solved for the unknown nodal displacements at node 2 and node 3 (node 1 is fixed).

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1/k_1 & 1/k_1 \\ 1/k_1 & 1/k_1 + 1/k_2 \end{bmatrix} \begin{bmatrix} P \\ P \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2P/k_1 \\ 2P/k_1 + P/k_2 \end{bmatrix} \qquad \text{B.1-12}$$

and the reaction force follows from Equation B.1-11, $F_1 = -k_1 u_2 = -2P$, which ensures equilibrium.

Note: The unconstrained system has 3 degrees-of-freedom (one displacement per node). The constrained system has only 2 degrees-of-freedom.

Now, with the knowledge of assembling and boundary conditions know, we can move onto more complex elements. The process of assembling a global stiffness matrix and applying boundary conditions are the same for any type of element, it is just the formulation (and size) of the stiffness matrix that may differ.

2.2 Stiffness Matrix Formulation: Bar Element

The bar element is similar to a spring element, i.e. it can only resist an axial force. Each bar element has 2 nodes, i.e. 2 degrees-of-freedom. Let's look at a simple bar element with constant area *A*, Young's modulus *E* and length *L*:

$$\sigma = E\varepsilon \;\rightarrow\; \varepsilon = \frac{\sigma}{E}, \qquad \sigma = \frac{F}{A} \;\rightarrow\; \varepsilon = \frac{F}{AE}$$

but also

$$\varepsilon = \frac{\Delta L}{L} \;\rightarrow\; \Delta L = \varepsilon L \;\rightarrow\; \Delta L = \frac{FL}{AE} \;\rightarrow\; F = \frac{AE}{L}\Delta L$$

Comparing this to a spring element,

$$F = k\Delta L \quad \rightarrow k = \frac{AE}{L} \qquad \text{B.2-1}$$

2.2.1 The Direct Method

The direct method of calculating the stiffness matrix was used for the spring element. Above it has been shown that the stiffness of a bar element is given by Equation B.2-1. Making use of Equation B.1-2, the stiffness for the bar element follows as,

$$k_{bar} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{2.2-2}$$

This, of course, assumes small deformations and elastic materials.

2.2.1 The Formal Procedure

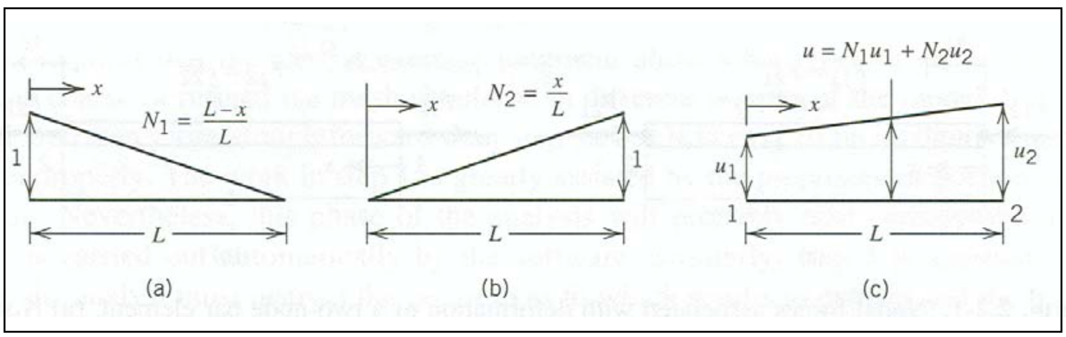


Figure 2.2-2 – Shape functions N_1 and N_2 for a two-node

Let's write an expression for the axial displacement u of an arbitrary point on the bar. Assume that node 2 is fixed, and node 1 is given a unit displacement, Figure 2.2-2a. The displacement at node 2 will be zero, and the displacement at node 1 should be unity (1). The deformation, for example, in the centre of the bar should be 0.5. This is written as

$$u = \frac{L-x}{L}u_1 \qquad \text{B.2-2}$$

where x starts from node 1 as in the figure, and u_1 is the displacement of node 1. The same can be done with node 1 fixed and node 2 given a unit displacement,

$$u = \frac{x}{L}u_1 \qquad \text{B.2-3}$$

Combining Equation B.2-2 and B.2-3 in matrix notation

$$u = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{or} \quad u = \underline{\overline{N}}\ \overline{d} \qquad \text{2.2-5}$$

where $\underline{\overline{N}}$ is called the shape function matrix and $\overline{d}$ the vector of element nodal degrees-of-freedom (translation in this case, but could also be rotations as in beam elements).

Let's now return to the definition of strain given in Chapter 1 (notes). Since this is a one-dimensional case (only axial deformation), there is only one strain component namely

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \text{B.2-4}$$

With *u* given by Equation 2.2-5,

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\underline{\overline{N}}\ \overline{d}\right],$$

but $\overline{d}$ is the nodal displacements (constants), and not a function of *x*,

$$\varepsilon_x = \frac{\partial u}{\partial x} = \left[\frac{\partial \underline{\overline{N}}}{\partial x}\right]\overline{d} = \underline{\overline{B}}\,\overline{d} \qquad \text{2.2-6}$$

where the matrix $\underline{\overline{B}}$ is called the *strain-displacement* matrix, containing shape function gradients, $\underline{\overline{B}} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$. **The bar element has constant strain**.

To formulate the stiffness matrix, we need to look at energy principles (**see section 3.1 of the textbook**). First assume a linear elastic spring/bar, with the force-displacement behavior as in Figure B.2-1.

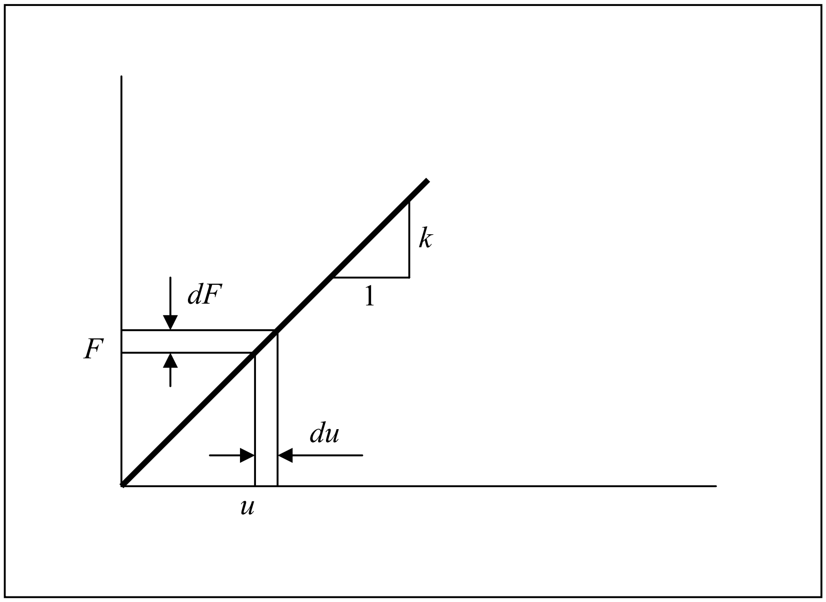


Figure B.2-1 – Force –displacement relation for simple spring/bar element

The increment in *work* done on the bar is given by

$$dW = F\,du$$

but $\quad F = k\,u \;\rightarrow dW = k\,u\,du \rightarrow W = \int k\,u\,du = \frac{1}{2}k\,u^2 = \frac{1}{2}F\,u \qquad$ B.2-5

This is the total *work* done on the bar when extended by a distance *u*. In the case of multi degrees-of-freedom, the *work* is given by

$$W = \frac{1}{2}\;\bar{d}^T\;\bar{F} \qquad \text{B.2-6}$$

where $\bar{d}$ is the vector containing nodal displacements *u.* The work done on the bar, Equation B.2-6, goes into increasing the *strain energy*. The increment in strain energy per unit volume is given by

$$dE_v = \sigma\,d\varepsilon$$

but $\quad \sigma = E\,\varepsilon = E\,\underline{\bar{B}}\,\bar{d}$ (using Equation 2.2-6)

thus $\quad E_v = \int\;\sigma\,d\varepsilon = \int E\,\varepsilon\;\;d\varepsilon = \frac{1}{2}E\,\varepsilon^2$

In the case of multi degrees-of-freedom, this becomes

$$E_v = \frac{1}{2}\bar{\varepsilon}^T E\,\bar{\varepsilon}$$

The total strain energy is calculated over the total volume of the element,

$$E = \int_V \frac{1}{2} \bar{\varepsilon}^T E \bar{\varepsilon} \; dV \qquad \text{B.2-7}$$

Making use of Equation 2.2-6

$$\begin{aligned} E = \int_V \frac{1}{2} \left(\underline{\bar{B}}\,\bar{d}\right)^T E \left(\underline{\bar{B}}\,\bar{d}\right) dV &= \int_V \frac{1}{2} \; \bar{d}^T \, \underline{\bar{B}}^T E \; \underline{\bar{B}} \; \bar{d} \; dV \\ &= \frac{1}{2} \bar{d}^T \left(\int_V \underline{\bar{B}}^T E \; \underline{\bar{B}} \;\; dV \right) \bar{d} \end{aligned} \qquad \text{B.2-8}$$

Conservation of energy states that the work (Equation B.2-6) input should equal the change in strain energy (Equation B.2-8),

$$\begin{aligned} W &= E \\ \therefore \frac{1}{2} \; \bar{d}^T \, \bar{F} &= \frac{1}{2} \bar{d}^T \left(\int_V \underline{\bar{B}}^T E \; \underline{\bar{B}} \;\; dV \right) \bar{d} \end{aligned} \qquad \text{B.2-9}$$

From this it is clear that

$$\bar{F} = \left(\int_V \underline{\bar{B}}^T E \; \underline{\bar{B}} \;\; dV \right) \bar{d} \qquad \text{B.2-10}$$

which is similar to the one-dimensional expression $F = k\,d$, and we conclude that

$$\underline{\bar{k}} = \int_V \underline{\bar{B}}^T E \; \underline{\bar{B}} \;\; dV \qquad 2.2\text{-}4$$

which is the general formulation for the stiffness matrix of an element.

Let's now apply this formulation to calculate the stiffness of the bar element. Using the definition of $\underline{\bar{B}}$ from Equation 2.2-6,

$$\underline{\bar{k}} = \int_L \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}^T E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \; A\,dx$$

where the cross section area A is taken to be constant.

$$\underline{\overline{k}} = \int_L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} A\,dx = E\,A \int_L \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} dx = AEL \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} \quad 2.2\text{-}7$$

$$= AE \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \\ -\frac{1}{L} & \frac{1}{L} \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

This result is the same as obtained using the direct method, Equation 2.2-2.

Note: The stiffness matrix formulation given in Equation 2.2-4 can be applied to **any** element. For different elements it would only be the strain-displacement matrix $\underline{\overline{B}}$ that differs and for two-dimensional and three-dimensional problems E is replaced by a matrix containing both E and Poisson's ratio v.

The process of assembly and constraints (reduced stiffness matrix) developed for spring elements can also be applied to the bar elements developed in this section. It is just the formulation of the element stiffness matrix that is different (although the same size 2 x 2).

2.3 Stiffness Matrix Formulation: Beam Element

2.3.1 The Direct Method: Simple Plane Beam

Figure 2.3-1a shows a simple plane beam element. The elastic modulus E and moment of inertia I are constant along the length of the beam. Only lateral displacements are assumed $v = v(x)$. For a beam only loaded at the end nodes with forces and moments, it can be shown from elementary beam theory that the lateral displacement is a cubic function in x. Each of the two nodes has three degrees-of-freedom, two translational and one rotational.

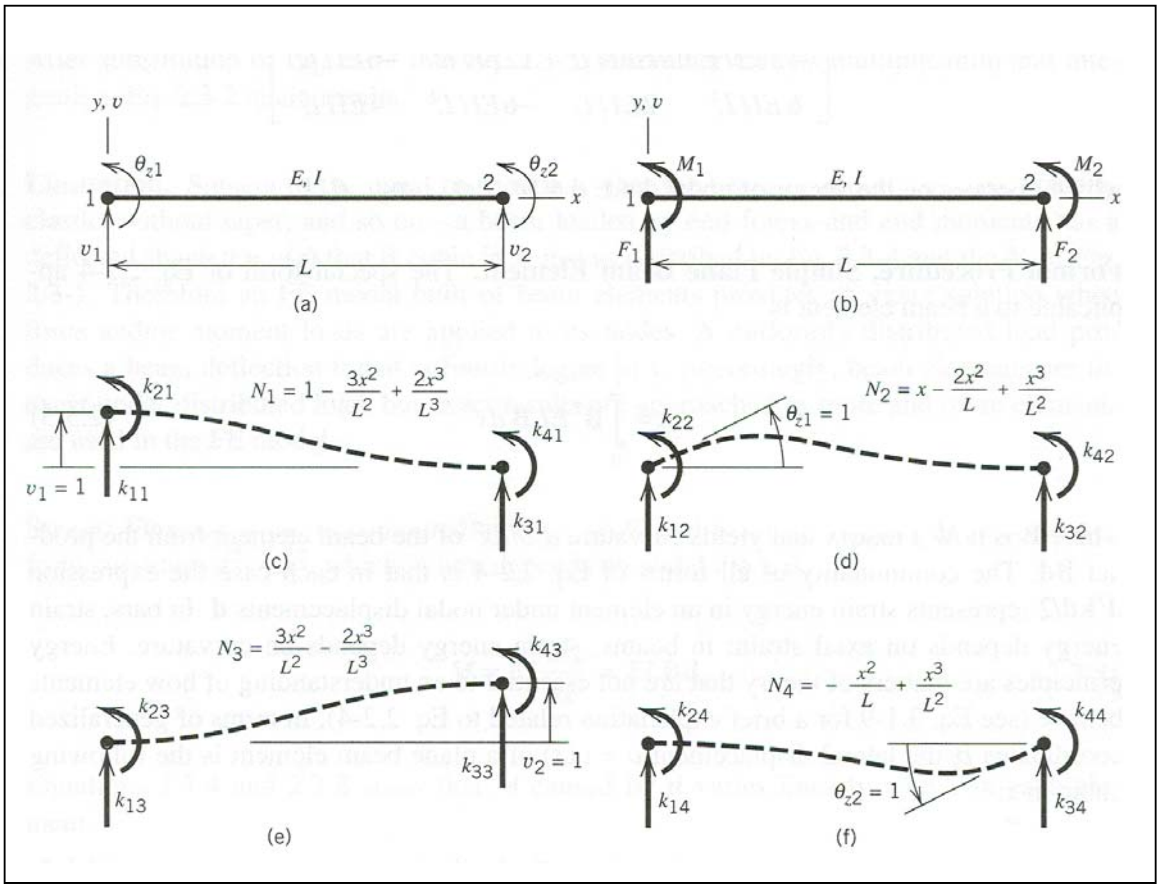


Figure 2.3-1 – Simple plane beam element and its nodal dof and shape function definitions

When the stiffness matrix for the bar element was developed, it was shown that the j^{th} column in the stiffness matrix represents the nodal forces (at each of the nodes respectively) that should be applied to have unit displacement (rotation) at the j^{th} degree-of-freedom and zero displacement (rotation) at all the other degrees-of-freedom. This procedure will be followed to calculate the beam stiffness matrix. Assume the degrees-of-freedom are assembled as $d = [v_1 \quad \theta_1 \quad v_2 \quad \theta_2]^T$.

The first degree-of-freedom is v_1. Set $v_1 = 1$ and all other degrees-of-freedom equal to zero. This is indicated in Figure 2.3-1c. Forces and moments should be applied at all degrees-of-freedom. What should these forces and moment be? Let's return to Basic beam theory[2] as indicated in Figure B.3-1.

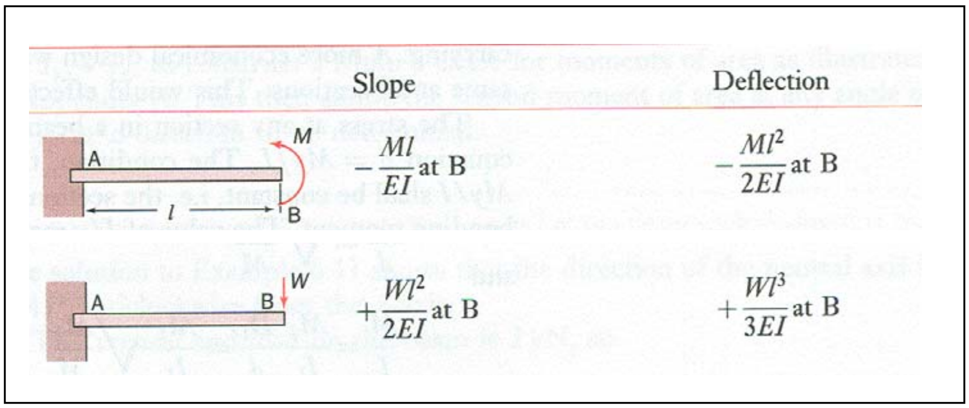


Figure B.3-1 – Basic beam theory

Using Figure B.3-1, the lateral displacement of node 1 (Figure 2.3-1c) can be written in terms of the applied moment and force at node 1:

$$v_1 = 1 = \frac{k_{11}L^3}{3EI} - \frac{k_{21}L^2}{2EI}, \qquad \text{B.3-1}$$

the slope (angle) at node 1 should also be zero:

$$\theta_1 = 0 = -\frac{k_{11}L^2}{2EI} + \frac{k_{21}L}{EI} \qquad \text{B.3-2}$$

[2] Benham, PP, Crawford, RJ, Armstrong, CG, *Mechanics of Engineering Materials*, Second Edition, 1996, Addison Wesley Longman Limited.

Solving these two equations simultaneously for k_{11} and k_{21}

$$k_{11} = \frac{12EI}{L^3} \qquad k_{21} = \frac{6EI}{L^2} \tag{B.3-3}$$

From equilibrium, we can write the following two equations,

Sum of forces in y-direction: $\quad 0 = k_{11} + k_{31}$

Sum of moments around node 2: $\quad 0 = k_{21} + k_{41} - k_{11}L$ B.3-4

Using Equation B.3-3, these can be solved:

$$k_{31} = -\frac{12EI}{L^3} \qquad k_{41} = \frac{6EI}{L^2} \tag{B.3-5}$$

Following this procedure for all the other degrees-of-freedom, the final element stiffness matrix is obtained,

$$k_{beam} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \tag{2.3-2}$$

Note: This element has 4 degrees-of-freedom, two per node.

The equilibrium equation for this element

$$\begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{bmatrix} \tag{B.3-6}$$

2.3.2 The Formal Procedure: Simple Plane Beam

In the case of a beam element the stiffness matrix is given by an expression similar to that in Equation 2.2-4,

$$\underline{\bar{k}} = \int_V \underline{\bar{B}}^T E\, I\, \underline{\bar{B}} \quad dV \qquad 2.3\text{-}3$$

where $\underline{\bar{B}}$ is now the matrix given the relation between nodal displacement and beam curvature $\frac{\partial^2 v}{\partial x^2}$. The lateral displacement of the beam is given in terms of generalized coordinates β_i

$$v = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3 \qquad 2.3\text{-}4$$

The β's can be written in terms of nodal displacements/rotations making use of a similar procedure used for bar elements. For example at $x = 0$, $v = v_1$ and $\theta = \theta_1 = \left.\frac{dv}{dx}\right|_{x=0}$.

The result is given by

$$v = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \underline{N}\,\bar{d} \qquad 2.3\text{-}5$$

The shape functions are given in Figure 2.3-1. The curvature of the beam element,

$$\frac{d^2 v}{dx^2} = \left[\frac{d^2}{dx^2}\underline{N}\right]\bar{d} = \underline{B}\,\bar{d} \qquad 2.3\text{-}6$$

where the strain-displacement matrix is (using the shape function definitions)

$$\underline{B} = \left[-\frac{6}{L^2} + \frac{12x}{L^3} \quad -\frac{4}{L} + \frac{6x}{L^2} \quad \frac{6}{L^2} - \frac{12x}{L^3} \quad -\frac{2}{L} + \frac{6x}{L^2} \right] \qquad 2.3\text{-}7$$

After substitution of Equation 2.3-7 into 2.3-3, and some manipulation, the same stiffness matrix as in Equation 2.3-2 is obtained.

This formulation gives an exact solution to the beam problem under the following conditions:

- The beam is initially straight
- Linear elasticity assumed
- No taper – constant cross section
- Under small deformations

The stress in the beam is given by the well-known formula $\sigma = \frac{My}{I}$ where the moment M is given by

$$M = EI\frac{d^2 v}{dx^2} = EI\underline{B}\,\overline{d} \qquad 2.3\text{-}8$$

2.3.2 2D Beam

A 2D beam is a combination of a bar element and a simple plane beam element. It can resist axial loads, transverse loads and bending moments. The equilibrium equation (and stiffness matrix) is a combination of that of the bar element (Equation 2.2-7) and that of the simple plane beam (Equation B.3-6)

$$\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F_{axial1} \\ F_{trans1} \\ M_1 \\ F_{axial2} \\ F_{trans2} \\ M_2 \end{bmatrix} \qquad B.3\text{-}7$$

2.4 Properties of the Stiffness Matrix: Avoiding Singularity

The element stiffness matrix $\underline{\bar{k}}$ and the global stiffness matrix $\underline{\bar{K}}$ are symmetric. This is, however, only true if the material behaviour is linear. The diagonal components of both stiffness matrices are always positive.

If a structure is unsupported or inadequately supported, $\underline{\bar{K}}$ can be singular, and the system equations unable to be solved. To prevent this, the structure must be supported (constrained) to avoid rigid body motion. A structure may also have a singular $\underline{\bar{K}}$ if it contains a mechanism.

2.5 Mechanical Loads: Stresses

Loads can be either applied forces or moments at a point, or surface pressure, or body forces (inertia, gravity). Concentrated forces and moment are applied at nodes (point loads). Moments, however, can only be applied at the node if at least one element connected to this node, has a rotational degree of freedom, e.g. a beam element.

Distributed loads, such as pressure, acts on elements between the nodes. These distributed loads must be converted to equivalent nodal loads, since in FEM, loads can only be applied at nodes. For bar elements, the following procedure can be used to obtain *work-equivalent loads*.

Take a distributed load q [N/m], acting on a bar element as indicated in Figure 2.5-1a. The load q causes the bar to extend. The nodal displacements would be u_1 and u_2. The *work* done by the distributed load is given by force times displacement. With the nodal displacements known, the displacements within the element follow from the shape functions, $u(x) = \underline{d}\,\overline{N}$. The *work* is given by $\int_L u(x)\,q(x)\;dx = \int_L \underline{d}\,\overline{N}\,q(x)\;dx$. We are now looking for nodal forces F_e that would result in the same amount of *work* being done - *work-equivalent nodal loads*.

$$\underline{d}\ \overline{F}_e = \int_L \underline{d}\,\overline{N}\,q(x)\ dx \qquad \text{B.5-1}$$

but since $\underline{d}$ are constant nodal displacements

$$\underline{d}\ \overline{F}_e = \underline{d}\left(\int_L \overline{N}\,q(x)\ dx\right) \quad \rightarrow \quad \overline{F}_e = \int_L \overline{N}\ q(x)\ dx \qquad \text{B.5-2}$$

In the case of a point load F, it becomes

$$\underline{d}\ \overline{F}_e = \int_L \underline{d}\,\overline{N}\,F\ dx \quad \rightarrow \overline{F}_e = \overline{N}\,F \qquad \text{B.5-3}$$

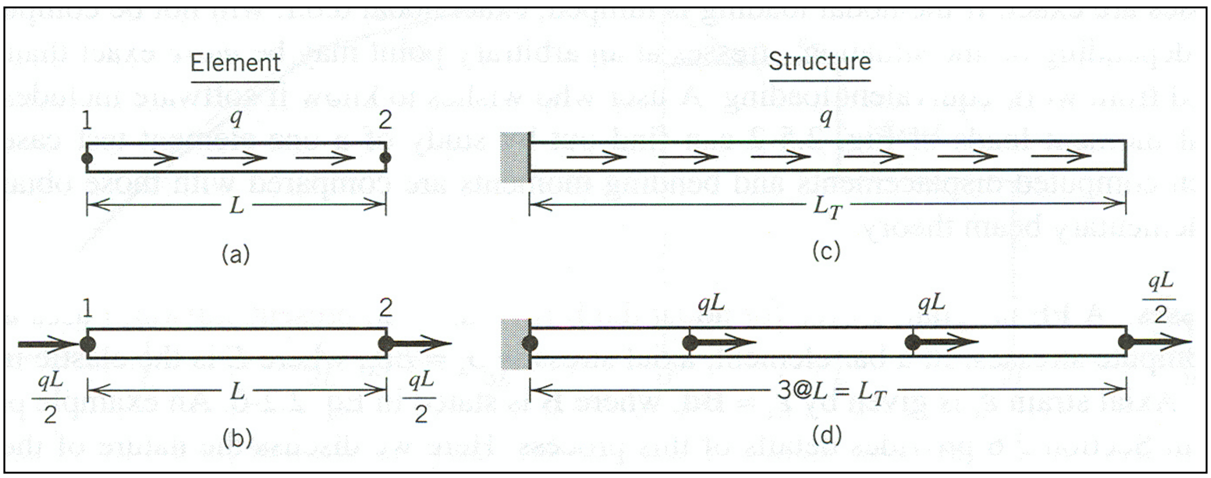


Figure 2.5-1 – Uniformed distributed axial force q on a two node bar element.

The following **example** demonstrates the procedure where a distributed and a point load is applied to a single bar element, Figure B.5-1. The distributed load q varies linearly from q_1 at node 1 to q_2 to at node 2,

$$q(x) = \frac{L-x}{L}q_1 + \frac{x}{L}q_2 \qquad \text{B.5-4}$$

and a point load F is applied at a distance $x = \frac{2L}{3}$. The *work equivalent nodal forces* are calculated using Equation B.5-2 and B.5-3

$$\overline{F}_e = \int_L \overline{N}\ q(x)\ dx + \overline{N}\,F \qquad \text{B.5-5}$$

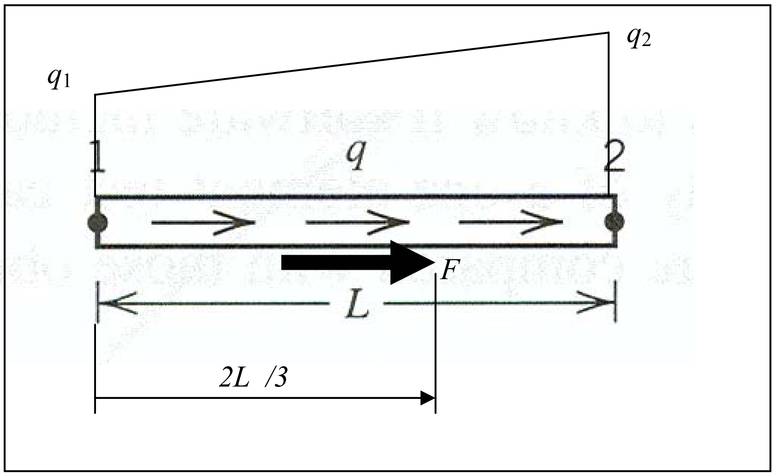


Figure B.5-1 – Equivalent nodal loads for a single bar element

Making use of the shape function definition of a bar element, Equation 2.2-5 $\overline{N} = \begin{bmatrix} \frac{L-x}{L} \\ \frac{x}{L} \end{bmatrix}$,

Equation B.5-5 becomes

$$\overline{F}_e = \int_L \begin{bmatrix} \frac{L-x}{L} \\ \frac{x}{L} \end{bmatrix} \left(\frac{L-x}{L} q_1 + \frac{x}{L} q_2 \right) dx + \begin{bmatrix} \frac{L-x}{L} \\ \frac{x}{L} \end{bmatrix}_{x=\frac{2L}{3}} F$$

$$\overline{F}_e = \int_L \begin{bmatrix} \frac{L-x}{L} \\ \frac{x}{L} \end{bmatrix} \left(\frac{L-x}{L} q_1 + \frac{x}{L} q_2 \right) dx + \begin{bmatrix} \frac{L-\frac{2L}{3}}{L} \\ \frac{\frac{2L}{3}}{L} \end{bmatrix} F$$

$$\overline{F}_e = \frac{L}{6} \begin{bmatrix} 2q_1 + q_2 \\ q_1 + 2q_2 \end{bmatrix} + \begin{bmatrix} F/3 \\ 2F/3 \end{bmatrix} \qquad \text{B.5-6}$$

In the case where the load is uniformly distributed, $q_1 = q_2 = q$, and $F = 0$,

$$\overline{F}_e = \begin{bmatrix} qL/2 \\ qL/2 \end{bmatrix} \qquad \text{B.5-7}$$

This is demonstrated in Figure 2.5-1b, where the total force is qL. Figure 2.5-1c and d shows how the nodal loads add together when more than one element is assembled.

2.6 Thermal Loads: Stresses

When the temperature of an unrestrained body is uniformly increased, the body will deform, but the stresses will be unchanged. Temperature gradients are, however, more complicated, and thermal stresses can occur even if the body is unrestrained. In this course only a uniform temperature change will be treated. Software does these calculations automatically, with the user only specifying the temperature change. The next example, however, shows how the thermal changes are handled.

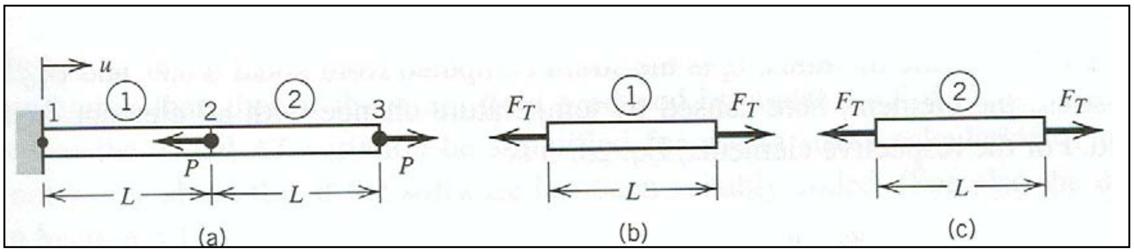


Figure 2.5-1 – Two element bar model loaded by externally applied load *P* and by uniform heating an amount ΔT

Figure 2.5-1a shows two bar elements assembled together. A load *P* is applied at node 2 in the negative *x*-direction, and a load *P* at node 3 in the positive *x*-direction. Let the cross sectional area *A*, elasticity *E* and the coefficient of thermal expansion α be constant over the bar.

Due to the temperature change alone, the bar will extend, but the stress will be unchanged. The change in strain can be modelled by applying nodal loads, but these loads will also cause the stress to increase. The solution is to apply *thermal loads*, that should yield the same strain deformation as the temperature change, and then to subtract the stress caused by the *thermal load*.

The definition of the coefficient of thermal extension $\varepsilon = \alpha \, \Delta T$ is used to calculate the *thermal loads*,

$$F_T = \sigma A = E\,\varepsilon A = \alpha\, A\, E\, \Delta T \qquad \text{B.6-1}$$

This load would cause the same increase/decrease in strain as the increase/decrease in temperature ΔT. The element stiffness matrix is given by

$$\underline{\bar{k}}_1 = \underline{\bar{k}}_2 = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and the thermal load } \bar{r}_{T1} = \bar{r}_{T2} = \begin{bmatrix} -F_T \\ F_T \end{bmatrix} \qquad 2.6\text{-}1$$

Assembly of elements yields the global equilibrium equation

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -F_T + R_1 \\ F_T - F_T - P \\ F_T + P \end{bmatrix} \qquad 2.6\text{-}2$$

where R_1 is the support reaction at node 1. The stiffness matrix in Equation 2.6-2 is singular. Singularity is removed by the support conditions, i.e. removing row 1 and column 1.

$$\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -P \\ F_T + P \end{bmatrix} \qquad 2.6\text{-}3$$

Solving this for the nodal displacements, and making use of Equation B.6-1,

$$u_1 = 0, \quad u_2 = \alpha L \Delta T, \quad u_3 = 2\alpha L \Delta T + \frac{PL}{AE} \qquad 2.6\text{-}4$$

To calculate the stress in each of the elements, we have to remove the stress due to *thermal loads*.

$$\sigma_{x1} = E\varepsilon_x - \alpha E \Delta T = E\frac{u_2 - u_1}{L} - \alpha E \Delta T = 0 \qquad 2.6\text{-}6a$$

$$\sigma_{x2} = E\varepsilon_x - \alpha E \Delta T = E\frac{u_3 - u_2}{L} - \alpha E \Delta T = \frac{P}{A} \qquad 2.6\text{-}6b$$

These results can be checked through inspection!

2.7 Transformations

The textbook only handles transformations in **Chapter 4, section 4.3**. However, it fits better in Chapter 2 and is needed before a complete bar/beam example can be done.

We have to diffirentiate between *local coordinates* and *global coordinates*. The user defines the geometry of a FEM model in the global coordinate system *XYZ*. Software typically generates an element stiffness matrix in local coordinates *xyz*, then automatically converts to global coordinates for assembly of elements. Global and local systems may be parallel or even coincident.

The stiffness matrix of a element is most easily written in local coordinates, but it may be arbitrary orientated in global coordinates. Rather than formulate the element stiffness matrix in global coordinates, it is easier to transform the stiffness matrix to global coordinates. Transformation is carried out outomatically by software. Let's first look at the transformation of a single point in two-dimensional space.

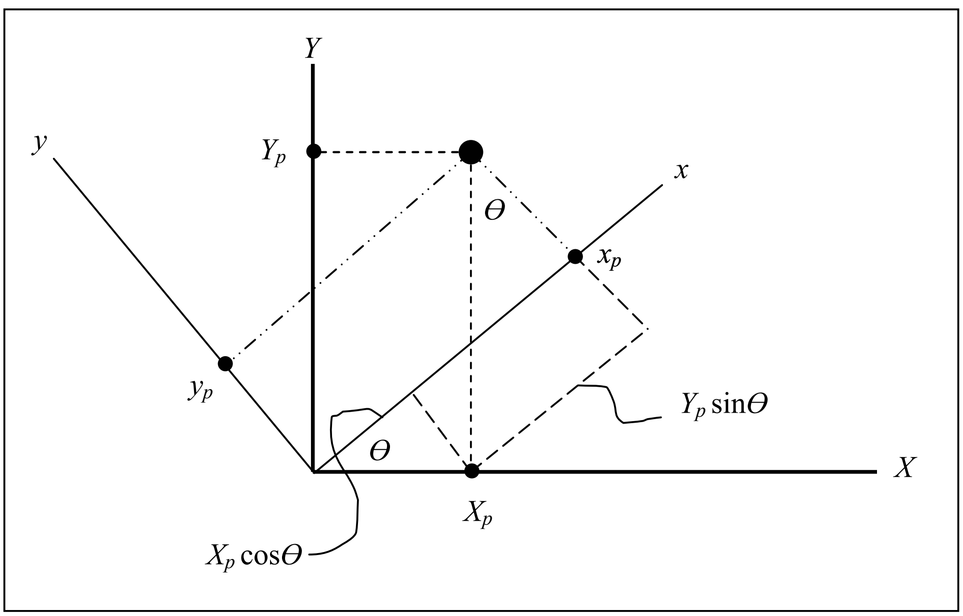


Figure B.7-1 – Coordinate transformation of a single point in two-dimensional space

From figure B.7-1, the following relations can be written,

$$x_p = X_p \cos\theta + Y_p \sin\theta$$
$$y_p = Y_p \cos\theta - X_p \sin\theta \qquad \text{B.7-1}$$

In matrix notation

$$\begin{bmatrix} x_p \\ y_p \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_p \\ Y_p \end{bmatrix} \qquad \text{B.7-2}$$

$$\therefore\ \bar{x}_p = \underline{\bar{\bar{T}}}\,\bar{X}_p$$

where $\underline{\bar{\bar{T}}}$ is called the *tranformation matrix* which is orthogonal: $\underline{\bar{\bar{T}}}^{-1} = \underline{\bar{\bar{T}}}^{T}$.

2.7.1 Bar Element

Let u'_1 and u'_2 be the nodal displacements in local coordinates of a bar element with two nodes and (u_1, v_1) and (u_2, v_2) the nodal displacements in global coordinates. The transformation becomes

$$\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \qquad \text{B.7-3}$$

or $\bar{d}' = \underline{\bar{T}}\,\bar{d}$ where $c = \cos\theta,\ s = \sin\theta$ and $\underline{\bar{T}}$ is build up from components in $\underline{\bar{\bar{T}}}$, Equation B.7-2. Similarly, the nodal forces can be written as

$$\bar{f}' = \underline{\bar{T}}\,\bar{f} \qquad \text{B.7-4}$$

The equilibrium equation in local coordinates

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} f'_{x1} \\ f'_{x2} \end{bmatrix}$$

$$\therefore \underline{\bar{k}}'\,\bar{d}' = \bar{f}' \qquad \text{B.7-5}$$

Using the transformations defined in Equation B.7-3 and B.7-4

$$\underline{\bar{k}}'\,\underline{\bar{T}}\,\bar{d} = \underline{\bar{T}}\,\bar{f} \qquad \text{B.7-6}$$

Multiplying both sides by $\underline{\overline{T}}^{T}$,

$$\underline{\overline{T}}^{T}\,\underline{\overline{k}}'\,\underline{\overline{T}}\,\overline{d} = \overline{f} \qquad \text{B.7-7}$$

where the force $\overline{f}$ is in global coordinates and the stiffness matrix in global coordinates is given by

$$\underline{\overline{T}}^{T}\,\underline{\overline{k}}'\,\underline{\overline{T}} = \frac{AE}{L}\begin{bmatrix} c^{2} & cs & -c^{2} & -cs \\ cs & s^{2} & -cs & -s^{2} \\ -c^{2} & -cs & c^{2} & cs \\ -cs & -s^{2} & cs & s^{2} \end{bmatrix} \qquad \text{B.7-8}$$

The element stress is calculated using

$$\sigma = E\varepsilon = E\,\underline{\overline{B}}\,\overline{d} = E\begin{bmatrix} \dfrac{-1}{L} & \dfrac{1}{L} \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$$

$$= E\begin{bmatrix} \dfrac{-1}{L} & \dfrac{1}{L} \end{bmatrix}\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \qquad \text{B.7-9}$$

$$= \frac{E}{L}\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

2.7.1 2D Beam Element

The equilibrium equation for a 2D beam element is given in Equation B.3-7,

$$\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_1' \\ v_1' \\ \theta_1' \\ u_2' \\ v_2' \\ \theta_2' \end{bmatrix} = \begin{bmatrix} F'_{axial1} \\ F'_{trans1} \\ M_1' \\ F'_{axial2} \\ F'_{trans2} \\ M_2' \end{bmatrix} \qquad \text{B.7-10}$$

Both the u and v displacements need to be rotated in the case of a local coordinate system not aligned with the global coordinate system. The rotational degree-of-freedom Θ, however, remains unchanged since the local and global z-axis (out-of plane) remains aligned no matter what the rotation (2D rotation). For a beam element, the transformation matrix can be written as

$$\underline{\overline{T}} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{B.7-11}$$

which is build up from components in $\underline{\overline{\overline{T}}}$, Equation B.7-2.

The following example demonstrates the use of coordinate transformation for bar elements.

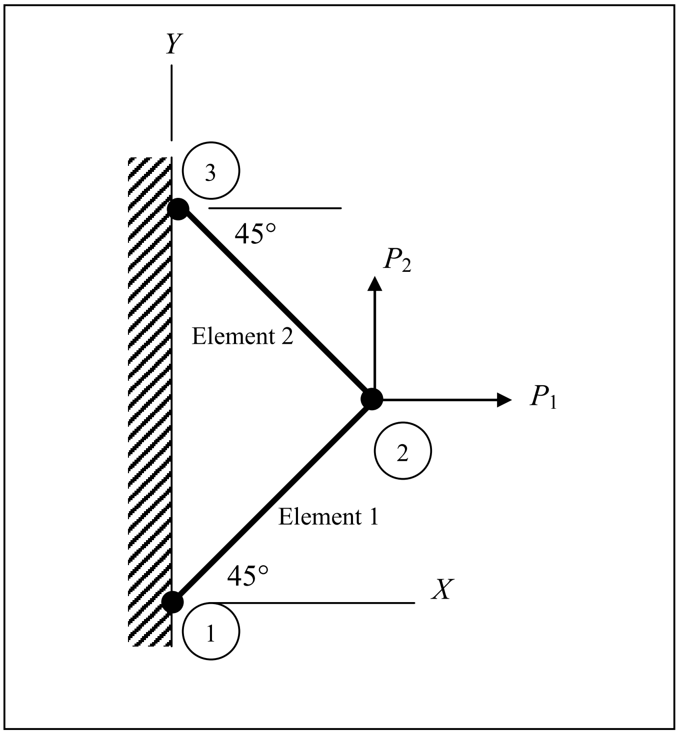


Figure B.7-2 – Example: Coordinate transformation

Figure B.7-2 shows two identical bar elements: A, E, length = L.

Find:

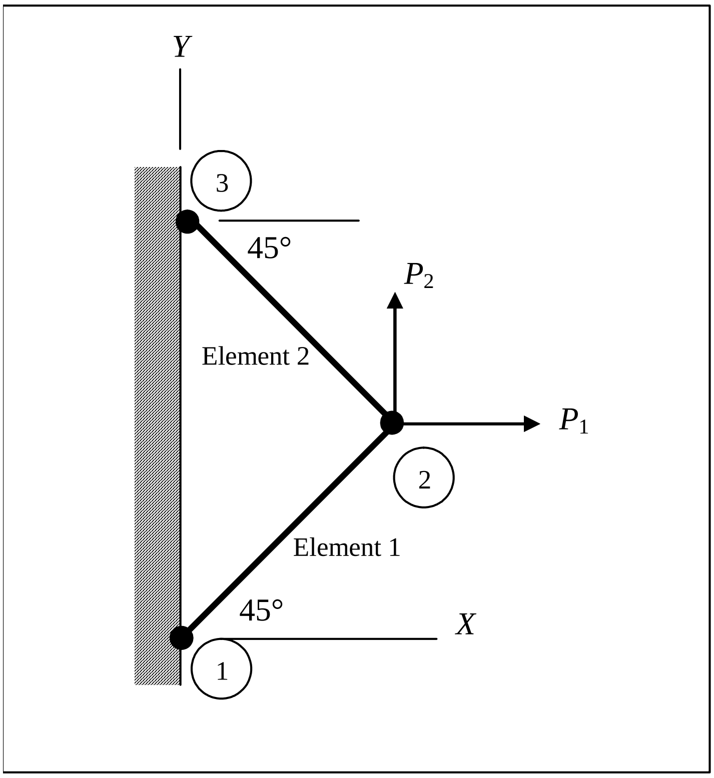
1) The unrotated stiffness matrix of each element.
2) The rotated stiffness matrix of each element.
3) The global unconstrained stiffness matrix.
4) The global constrained (reduced) stiffness matrix.
5) The global force vector.
6) The displacement of node 2.
7) The stress in each bar.

Example: Coordinate Transformation

$kPa := 1 \cdot 10^{3} Pa$ $\quad$ $kN := 1 \cdot 10^{3} N$

$MPa := 1 \cdot 10^{6} Pa$ $\quad$ $MN := 1 \cdot 10^{6} N$

PROBLEM:



Find 1) The unrotated stiffness matrix of each element.

2) The rotated stiffness matrix of each element

3) The global unconstrained stiffness matrix

4) The global constrained (reduced) stiffness matrix

5) The global force vector

6) The displacement of node 2

7) The stress in each bar.

Solution 1:

In local coordinates the stiffness matrix is:

$$k_1 = k_2 = \frac{A \cdot E}{L} \begin{array}{cc} \begin{array}{cc} u_1 & u_2 \end{array} & \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \begin{array}{c} u_1 \\ u_2 \end{array} \end{array}$$

Solution 2:

For element 1 the rotation angle is $\theta_1 := 45\text{deg}$

$\cos(\theta_1) = 0.707$ $\qquad \frac{\sqrt{2}}{2} = 0.707$

Using Equation B.7-8 (Lecture notes)

$$k_1 = \frac{A \cdot E}{2 \cdot L} \cdot \begin{array}{cc} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} & \text{Global numbers:} \\ \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} & \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array} \end{array}$$

For element 2 the rotation angle is $\theta_2 := 135\text{deg}$

$$k_2 = \frac{A \cdot E}{2 \cdot L} \cdot \begin{array}{cc} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} & \\ \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} & \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \end{array}$$

Solution 3:

$$\frac{A \cdot E}{2 \cdot L} \cdot \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{pmatrix}$$

Solution 4:

Node 1 and Node 3 are fully constrained in the x- and y-directions: Delete row and column and 5,6 respectively:

$$K_{reduced} = \frac{A \cdot E}{2 \cdot L} \cdot \begin{array}{cc} \begin{array}{cc} u_2 & v_2 \end{array} & \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{array}{c} u_2 \\ v_2 \end{array} \end{array}$$

Solution 5:

$$F_{reduced} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

Solution 6:

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \left[\frac{A \cdot E}{2 \cdot L} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right]^{-1} \cdot \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \frac{L}{A \cdot E} \cdot \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

Solution 7:

Using Equation B.7-9 (Lecture notes)

$$\sigma_1 = \frac{E}{L} \cdot \frac{\sqrt{2}}{2} \cdot \begin{pmatrix} -1 & -1 & 1 & 1 \end{pmatrix} \cdot \frac{L}{A \cdot E} \cdot \begin{pmatrix} 0 \\ 0 \\ P_1 \\ P_2 \end{pmatrix} = \frac{\sqrt{2}}{2 \cdot A} \cdot (P_1 + P_2)$$ for element 1

$$\sigma_2 = \frac{E}{L} \cdot \frac{\sqrt{2}}{2} \cdot \begin{pmatrix} -1 & -1 & 1 & 1 \end{pmatrix} \cdot \frac{L}{A \cdot E} \cdot \begin{pmatrix} P_1 \\ P_2 \\ 0 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2 \cdot A} \cdot (P_1 - P_2)$$ for element 2