Lecture 1

13 February 2003

Today we

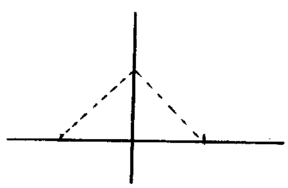
* even and odd functions
* periodic functions
* orthogonal functions

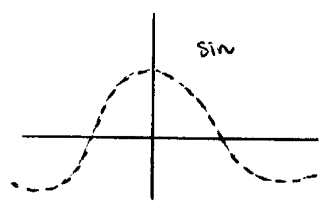
EVEN & ODD FUNCTIONS

An even function satisfies

$$f(x) = f(-x)$$

eg.

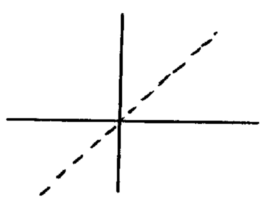


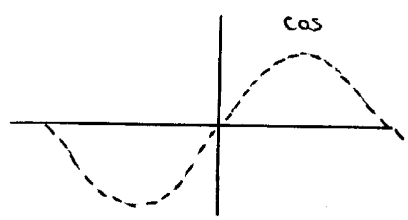


An odd function satisfies

$$f(x) = -f(-x)$$

eg.





One result which we are going to find useful

$$I = \int_{-A}^{A} f(x)\,dx$$

$$= \int_{-A}^{0} f(x)\,dx + \int_{0}^{A} f(x)\,dx$$

$$= \int_{0}^{A} f(-x)\,dx + \int_{0}^{A} f(x)\,dx$$

Make the substitution $s = -x$ in the 1st integral.

$$= \int_{0}^{A} \left[f(-x) + f(x) \right] dx$$

$$= \begin{cases} 0 & \text{if } f \text{ is odd} \\ 2\int_{0}^{A} f(x)\,dx & \text{if } f \text{ is even} \end{cases}$$

For any function $f(x)$

$$f(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right]$$

Convince yourself that this is true.

Note that

$g(x) = f(x) + f(-x)$ is an even function

since $g(-x) = f(-x) + f(x) = g(x)$,

and

$h(x) = f(x) - f(-x)$ is an odd function

since
$$h(-x) = f(-x) - f(x) = -\left[f(x) - f(-x)\right]$$
$$= -h(x).$$

So any function can be written as a sum of an even function & an odd function

$$f(x) = \underbrace{\frac{1}{2}\left[f(x) + f(-x)\right]}_{\text{even function}} + \underbrace{\frac{1}{2}\left[f(x) - f(-x)\right]}_{\text{odd function}}$$

eg:

$$e^x = \frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$= \cosh x + \sinh x .$$

Periodic Functions

A periodic function is a function which satisfies

$$f(x) = f(x+L) \quad \text{for all } x$$

then the period of f is L.

For a periodic function f with period L

$$\int_{x_0}^{x_0+L} f(x)\,dx = \int_0^L f(x)\,dx \quad \text{for all } x_0 .$$

The proof of this will be part of the first assigment.

ORTHOGONALITY

Two functions f and g are said to be orthogonal on the interval $[0,L]$ if

$$\int_0^L f(x)\,\overline{g(x)}\,dx = 0.$$

Recall that $\overline{g(x)}$ is the complex conjugate so if $g(x)$ is a real function we want

$$\int_0^L f(x)\,g(x)\,dx = 0.$$

eg: $f(x) = x$ $\qquad g(x) = x^4 - 3x^2 + 10$

over the interval $[-1,1]$

$$\int_{-1}^{1} f(x)\,g(x)\,dx = 0 \longleftarrow$$

This integral is easy if we use our new knowledge on even & odd functions.

So $f(x)$ and $g(x)$ are orthogonal over $[-1,1]$.

eg: $f(x) = \cos \frac{m\pi x}{\ell}$, $g(x) = \cos \frac{n\pi x}{\ell}$

over the interval $[-\ell, \ell]$ (m, n integers)

$$\int_{-\ell}^{\ell} \cos \frac{m\pi x}{\ell} \cos \frac{n\pi x}{\ell} \, dx$$

Use $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$= \int_{-\ell}^{\ell} \frac{1}{2} \left[\cos \frac{(m+n)\pi x}{\ell} + \cos \frac{(m-n)\pi x}{\ell} \right] dx$$

$$= \frac{1}{2} \frac{\ell}{(m+n)\pi} \sin \frac{(m+n)\pi x}{\ell} \Bigg|_{-\ell}^{\ell} + \frac{1}{2} \frac{\ell}{(m-n)\pi} \sin \frac{(m-n)\pi x}{\ell} \Bigg|_{-\ell}^{\ell}$$

$$= \begin{cases} 0, & m \neq n \\ \ell, & m = n \neq 0 \\ 2\ell, & m = n = 0 \end{cases}$$

Do for exercise

$$\int_{-\ell}^{\ell} \sin \frac{m\pi x}{\ell} \sin \frac{n\pi x}{\ell} \, dx = \begin{cases} 0, & m \neq n \\ \ell, & m = n \end{cases}$$

for $m, n \geq 1$.

Why don't we include $m = n = 0$?

$$\int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell} \cos\frac{m\pi x}{\ell}\, dx = 0$$

Once again this integral is easy if we use our knowledge of even & odd functions: $\underbrace{(\text{even function}) \times (\text{odd function})}_{\text{odd function.}}$

$\boxed{\text{So } f(x) \text{ and } g(x) \text{ are orthogonal for all integer } m \text{ and } n}$

We need these results in later work and also the following results:

eg:

$$f(x) = e^{\frac{2\pi i m x}{L}} \quad \text{and} \quad g(x) = e^{\frac{2\pi i n x}{L}}$$

Recall that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and the note that f and g have period L:

$$f(x+L) = f(x) \quad \text{and} \quad g(x+L) = g(x).$$

$$\int_0^L e^{\frac{2\pi i m x}{L}} \, e^{\frac{-2\pi i n x}{L}} \, dx$$

$$= \int_0^L e^{\frac{2\pi i (m-n)x}{L}} \, dx$$

$$= \frac{L}{2\pi i (m-n)} \, e^{\frac{2\pi i (m-n)x}{L}} \Bigg|_0^L$$

$$= \frac{L}{2\pi i (m-n)} \left[e^{2\pi i (m-n)} - 1 \right]$$

$$= \begin{cases} L, & m = n \\ 0, & m \neq n \end{cases}$$

[Use $e^{2\pi i (m-n)} = \cos 2\pi (m-n) + i \sin(2\pi)(m-n)$

So f and g are orthogonal of $[0, L]$.

Lecture 2: Fourier series expansion

Goal: Given a function of period 2ℓ, find its trigonometric series.

Given $f(x) = f(x+2\ell)$ find the Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right] \quad (*)$$

Note: The only common period of $\cos\frac{n\pi x}{\ell}$ and $\sin\frac{n\pi x}{\ell}$, $n=1,2,\dots$ is 2ℓ.

Calculating the coefficients (see Lecture notes):

This is based on the identities:

$$\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\cos\frac{n\pi x}{\ell}\,dx = \begin{cases} 2\ell, & m=n=0 \\ \ell, & m=n\neq 0 \\ 0, & m\neq n \end{cases}$$

$$\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\sin\frac{n\pi x}{\ell}\,dx = 0 \quad \text{vir alle } m,n$$

$$\int_{-\ell}^{\ell} \sin\frac{m\pi x}{\ell}\sin\frac{n\pi x}{\ell}\,dx = \begin{cases} 0, & m\neq n \\ \ell, & m=n \end{cases}$$

A little intuition concerning Fourier series of a functin $f(x)$

$$f(x) = \underbrace{a_0}_{\text{'average'}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}}_{\text{even function}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}}_{\text{odd function}}$$

The Fourier series automatically represents the function $f(x)$ as the sum of

an average, an even function and an odd function.

So it makes sense that if

$f(x)$ is even $\Rightarrow$ there is only an average and an even part to the Fourier series

$\Rightarrow$ $b_n = 0$, $n = 1, 2, \ldots$

$f(x)$ is odd $\Rightarrow$ there is only an average and an odd part to the Fourier series

$\Rightarrow$ $a_n = 0$, $n = 1, 2, \ldots$

Note: $n \neq 0$

$$\int_{-\ell}^{\ell} f(x)\,dx = a_0 \int_{-\ell}^{\ell} dx + \sum_{n=1}^{\infty} a_n \int_{-\ell}^{\ell} \cos\frac{n\pi x}{\ell}\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell}\,dx$$

$=0$ integral of odd function over symmetric interval

$$= 2\ell a_0$$

i.e.

$$\boxed{a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx}$$

Note: a_0 is the 'average' of $f(x)$.

This idea provides a good check on the calculation of a_0.

Multiply $(*)$ with $\cos\frac{m\pi x}{\ell}$, $m=1,\ldots$ and integrate

$$\int_{-\ell}^{\ell} f(x)\cos\frac{m\pi x}{\ell}\,dx = a_0\int_{-\ell}^{\ell}\cos\frac{m\pi x}{\ell}\,dx$$

$$+ \sum_{n=1}^{\infty}\left(a_n \int_{-\ell}^{\ell} \cos\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell}\,dx + b_n\int_{-\ell}^{\ell}\cos\frac{m\pi x}{\ell}\sin\frac{n\pi x}{\ell}\,dx\right)$$

$$= \ell a_m$$

i.e.

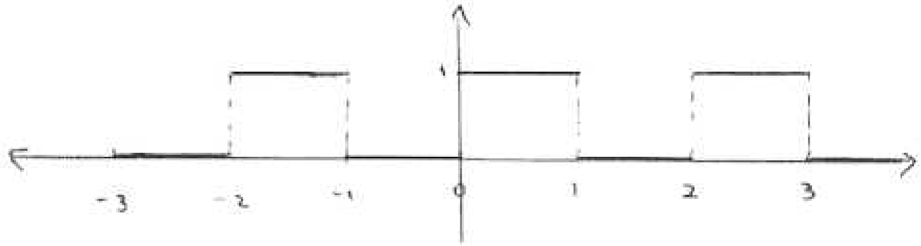
$$\boxed{a_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{m\pi x}{\ell}\,dx \qquad m=1,2,\ldots}$$

Similary, by multiplying (*) with $\sin \frac{m\pi x}{\ell}$, $m=1,2,\dots,$ and integrating we get

$$\boxed{b_m = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{m\pi x}{\ell}\,dx \qquad m=1,2,\dots}$$

Example 1:

$f(x) =$



Period $2 \Rightarrow \ell = 1$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_0^1 \cos n\pi x\,dx = \frac{1}{n\pi}\sin n\pi x\Big|_0^1$$

$$= \frac{1}{n\pi}\sin n\pi = 0$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_0^1 \sin n\pi x\,dx$$

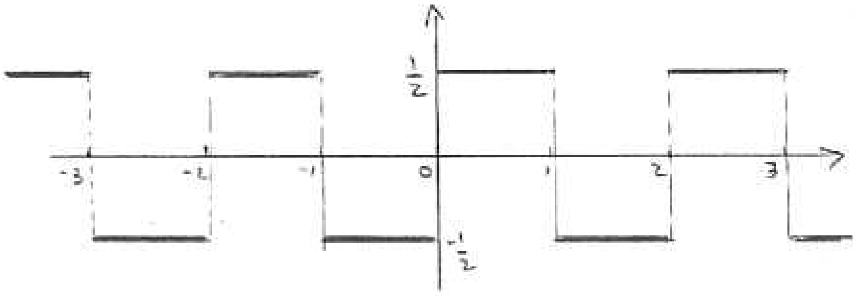
$$= -\frac{1}{n\pi}\left(\cos n\pi - 1\right).$$

Fourier series

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n\pi}\right)\sin n\pi x$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[2\sin x + \frac{3}{2}\sin 3x + \frac{2}{5}\sin 5x + \dots\right].$$

Example 2:

$f(x) =$ 

Period 2 $\Rightarrow$ $l = 1$

$f(x)$ is odd i.e. $a_n = 0$, $n = 0, 1, \ldots$

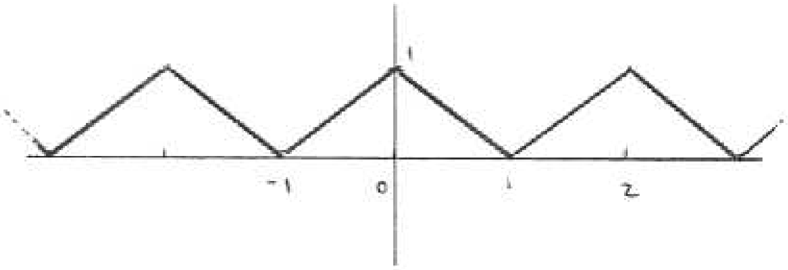
$$b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx$$

See figures on page 7.

$$= 2 \int_0^1 \frac{1}{2} \sin n\pi x \, dx = \frac{-1}{n\pi} (\cos n\pi - 1).$$

Question: Why is this the same as b_n in Example 1? Hint: Write $f(x)$ in example 1 as the sum of even and odd functions.

Example 3:

$f(x) =$ 

Period 2 $\Rightarrow$ $l = 1$

$f(x)$ is even $\Rightarrow$ $b_n = 0$, $n = 1, 2, \ldots$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\cdot 2\int_0^1 (1-x)\,dx = -\frac{1}{2}(1-x)^2\Big|_0^1$$
$$= \frac{1}{2}$$

Integration by parts: a technique we will use often in this course, so worth a bit of revision.

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx$$
$$= 2\int_0^1 (1-x)\cos n\pi x\,dx$$
$$= \frac{2}{n\pi}\Big[(1-x)\sin n\pi x\Big]\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx$$
$$= 0 + \frac{2}{n\pi}\left(\frac{1-\cos n\pi}{n\pi}\right) = \frac{2}{n^2\pi^2}(1-\cos n\pi).$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n^2}\right)\cos n\pi x$$
$$= \frac{1}{2} + \frac{4}{\pi^2}\left[\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \cdots\right]$$

Note: In this example the coefficients decay as $\frac{1}{n^2}$, while in previous examples it was $\frac{1}{n}$. Any ideas why?

<u>For your own practice</u>

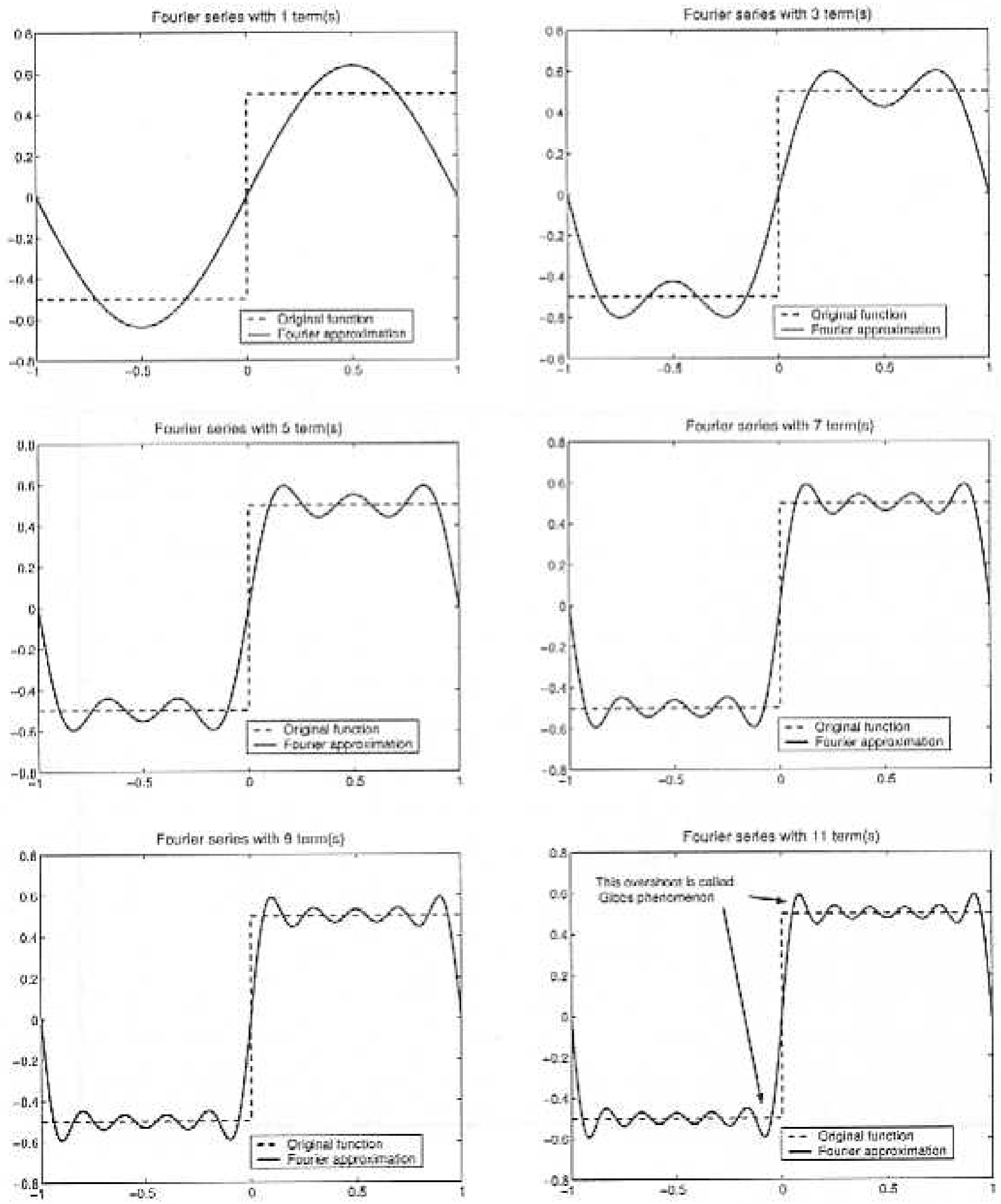
Calculate the Fourier series of f, given over one period as follows:

(a) $|x|$ on $(-2\pi, 2\pi]$

(b) $\cos^2 x$ on $(0,\pi]$ (c) e^{-x} on $[0,2)$.

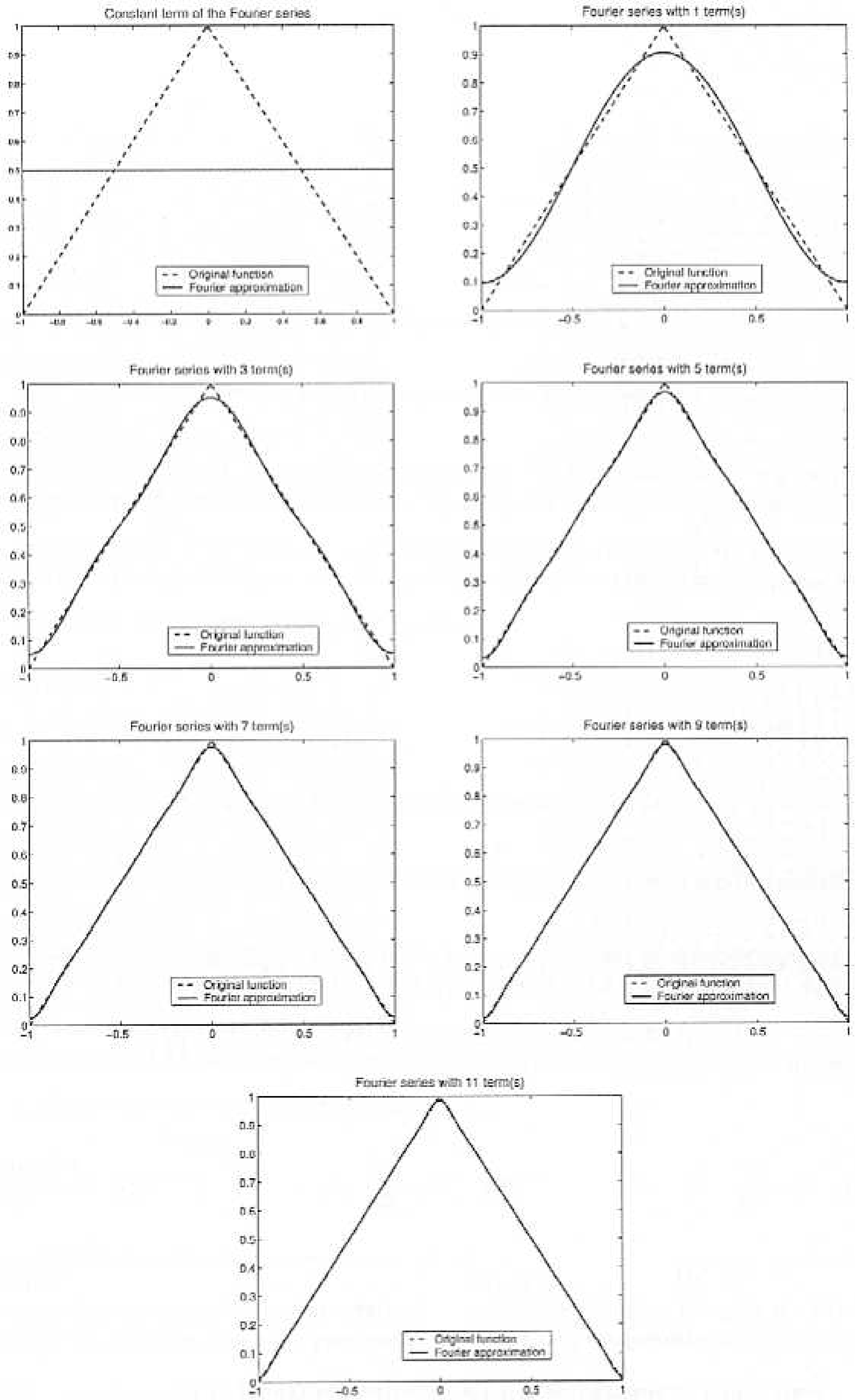
Lecture 2: Fourier series - illustration

Example 2



The function $f(x)$ and its Fourier series with n terms is plotted here over $[-1,1]$. Since both are periodic we know what they look like on the rest of the real line.

Example 3

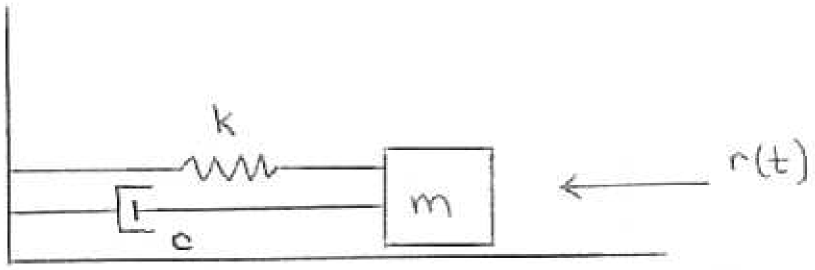


Again — we've just plotted $f(x)$ and its Fourier series over 1 period.

Lecture 3: Forced Oscillations

Goal: to develop an application of Fourier Series

A mass m is attached to a spring with spring constant k and a damper with constant c on a horisontal surface. An external force $r(t)$ is applied to the system. This system can be modelled as follows:



The differential equation which models this system is

$$m\ddot{x} + c\dot{x} + kx = r(t).$$

We assume the forcing term $r(t)$ is periodic. The solution to this differential equation has the form

$$x(t) = \underbrace{x_h(t)}_{\text{homogeneous solution}} + \underbrace{x_p(t)}_{\text{particular solution}}$$

<u>Question</u> : What happens to the homogeneous solution?

$$x_h(t) \to 0 \quad \text{as} \quad t \to \infty .$$

This can be verified by solving

$$m\ddot{x} + c\dot{x} + kx = 0$$

and noticing that $x_h(t) = e$

and therefore decays rapidly as $t \to \infty$.

This can also be justified physically as the homogenous solution of the differential equation describes the movement of the mass m when $r(t)=0$, that is there is no forcing term. So it makes sense that the mass comes to a standstill since there is a damper in the system.

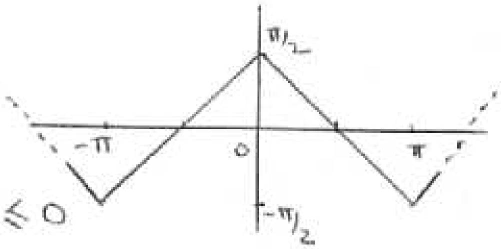
We are therefore interested in the particular solution to the differential equation, since the homogenous solution decays as $t \to \infty$.

Idea to solve this diff. equation:

Expand $r(t)$ in its Fourier series, then find the particular solution.

Example:

$$\ddot{x} + 0.02\dot{x} + 25x = r(t)$$



where
$$r(t) = \begin{cases} t + \pi/2, & -\pi < t \leq 0 \\ -t + \pi/2 & 0 < t \leq \pi \end{cases}$$

$r(t)$ is an even function

$$\Rightarrow \quad r(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} r(t)\,dt = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} r(t)\cos nt\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(-t+\pi/2\right)\cos nt\,dt$$

$$= \frac{2}{n\pi}\left(-t+\pi/2\right)\sin nt\Big|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}\sin nt\,dt$$

$$= \frac{2}{n^2\pi}\left(-\cos nt\right)\Big|_0^{\pi} = -\frac{2}{n^2\pi}\left(\cos n\pi - 1\right)$$

that is
$$r(t) = \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left(1-\cos n\pi\right)\cos nt$$

$$= \sum_{n=1,3,5,\dots}\frac{4}{n^2\pi}\cos nt$$

Each term in this expansion of $r(t)$ contributes one term to the solution of the equation, so we need to solve for x_1 in

$$\ddot{x}_1 + 0.02\dot{x}_1 + 25x_1 = \frac{4}{\pi}\cos t$$

and x_2 in

$$\ddot{x}_2 + 0.02\dot{x}_2 + 25x_2 = \frac{4}{2^2\pi}\cos 2t$$

and x_3 in

$$\ddot{x}_3 + 0.02\dot{x}_3 + 25x_3 = \frac{4}{3^2\pi}\cos 3t$$

Pick a generic term $\frac{4}{n^2\pi}\cos nt$ and try and solve

$$\ddot{x} + 0.02\dot{x} + 25x = \frac{4}{n^2\pi}\cos nt$$

Guess: $\quad x_n = A_n \cos nt + B_n \sin nt$

then $\quad \dot{x}_n = -nA_n \sin nt + nB_n \cos nt$

$$\ddot{x}_n = -n^2 A_n \cos nt - n^2 B_n \sin nt$$

Some motivation for this guess:

We want a function x_n so that a combination of x_n, its first derivative $\dot{x}_n$ and second derivative $\ddot{x}_n$ is equal to $\cos nt$. So by guessing x_n to be a combination of cosine and sine functions we achieve this.

Substitute into the diff. equation

$$\left[-n^2 A_n \cos nt - n^2 B_n \sin nt\right] + 0.02\left[-nA_n \sin nt + nB_n \cos nt\right]$$

$$+ 25\left[A_n \cos nt + B_n \sin nt\right] = \frac{4}{n^2\pi}\cos nt$$

This should to hold for all t, so we can gather coefficients of $\cos nt$ and $\sin nt$: (6)

$$\cos nt : \quad -n^2 A_n + 0.02n B_n + 25 A_n = \frac{4}{n^2 \pi}$$

$$(25 - n^2) A_n + 0.02n B_n = \frac{4}{n^2 \pi} \quad \cdots (1)$$

$$\sin nt : \quad -n^2 B_n - 0.02n A_n + 25 B_n = 0$$

$$-0.02n A_n + (25 - n^2) B_n = 0 \quad \cdots (2)$$

two equations and two unknowns, solve.

then $B_n = \dfrac{0.08}{n\pi Q_n}$ with $Q_n = (25-n^2)^2 + (0.02n)^2$

$A_n = \dfrac{4(25-n^2)}{n^2 \pi Q_n}$

---

so $$x_n = \frac{4(25-n^2)}{n^2 \pi Q_n} \cos nt + \frac{0.08}{n\pi Q_n} \sin nt$$

With these values for A_n and B_n, the particular solution to the given diff. equation is

$$x_p(t) = \sum_{n=1,3,5,\dots} \left(A_n \cos nt + B_n \sin nt \right)$$

The particular solution consists of many frequencies.

Is there a dominant one?

The amplitude of x_n is

$$R_n = \sqrt{A_n^2 + B_n^2}.$$

Calculate some specific values for the example above

$$R_1 = 0.0531$$

$$R_3 = 0.0088$$

$$R_5 = 0.5093 \longrightarrow \text{largest one}$$

$$R_7 = 0.0011$$

Dominant term of particular solution is x_5

Period of x_5 is $\frac{2\pi}{5}$

Recall that the period of the forcing term is $2\pi \neq \frac{2\pi}{5}$

$\frac{2\pi}{5}$ is the natural frequency of this system, dependent on the spring constant $k=5$.

Lecture 4: Complex Fourier Series

Aim: 1. To make life easier by getting rid of the sines and co-sines.

2. Extend the interval to get the desired Fourier series.

Need to know: $e^{i\theta} = \cos\theta + i\sin\theta$

This implies $e^{i2\pi n} = \cos 2\pi n + i\sin 2\pi n = 1$, $n \in \mathbb{Z}$, $n \neq 0$

Exercise: show that (a) $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

(b) $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

Let $f(x) = f(x+L)$

Note: (a) Period is L not $2l$

(b) The common period of $e^{2\pi i n x/L}$, $n \in \mathbb{Z}$ is L. (Prove it!)

Instead of sines and cosines, write

$$f(x) = \sum_{n=-\infty}^{\infty}{}' a_n e^{2\pi i n x/L}$$ Complex Fourier series

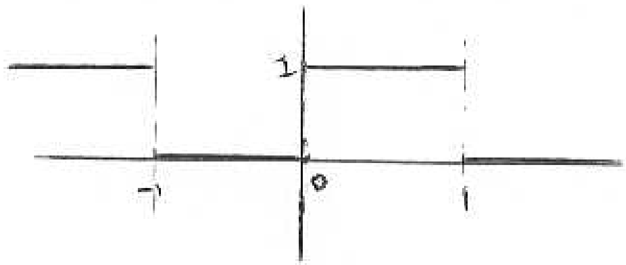
Multiply with $e^{-2\pi i m x/L}$ and integrate

$$\int_0^L f(x)\, e^{-2\pi i m x/L}\, dx = \sum_{n=-\infty}^{\infty} a_n \int_0^L e^{2\pi i (n-m)x/L}\, dx$$

Since $\displaystyle\int_0^L e^{2\pi i(n-m)x/L}\, dx = \begin{cases} L & \text{if } n=m \\ 0 & \text{if } n \neq m \end{cases}$

$$\boxed{a_m = \frac{1}{L}\int_0^L f(x)\, e^{-2\pi i m x/L}\, dx \,, \quad m \in \mathbb{Z}}$$

Example: $f(x) =$



$L=2 \Rightarrow$ Complex Fourier Series

is $\displaystyle f(x) = \sum_{n=-\infty}^{\infty} a_n e^{i\pi n x}$

with

$$a_n = \frac{1}{2}\int_0^2 f(x)\, e^{-2\pi i n x/2}\, dx$$

$$= \frac{1}{2}\int_0^1 e^{-i\pi n x}\, dx = \frac{-1}{2\pi i n} e^{-i\pi n x}\Big|_0^1 \quad (n \neq 0)$$

$$= \frac{-1}{2\pi i n}\left(e^{-i\pi n} - 1\right)$$

$$= \frac{-1}{2\pi i n}\left(\cos n\pi - 1\right)$$

$$= \frac{-1}{2\pi i n}\left((-1)^n - 1\right) = \begin{cases} \frac{1}{\pi i n}, & n \text{ odd} \\ 0, & n \text{ even} \\ & n \neq 0 \end{cases}$$

$$a_0 = \frac{1}{2} \qquad \Rightarrow \qquad a_n = \begin{cases} \frac{1}{\pi i n} & n \text{ odd} \\ 1/2 & n = 0 \\ 0 & n \text{ even}, \ n \neq 0 \end{cases}$$

i.e.

$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left(e^{i\pi x} - e^{-i\pi x}\right) + \frac{1}{i3\pi}\left(e^{i3\pi x} - e^{-i3\pi x}\right)$$

$$+ \frac{1}{i5\pi}\left(e^{i5\pi x} - e^{-i5\pi x}\right) + \cdots$$

This is were we usually stop, but we want to compare this to the Fouries series we obtained before (Lecture 2)

$$f(x) = \frac{1}{2} + \frac{1}{i\pi}\left(2i\sin x\right) + \frac{1}{i3\pi}\left(2i\sin 3x\right) + \cdots$$

using (b)

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \cdots$$

this is the same as before!

Suppose we know $f(x)$ only over the interval $(0, \ell)$.
We have many choices to calculate the Fourier series of f.

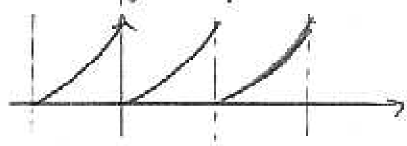
<u>Note</u>: We only want to represent $f(x)$ over $(0, \ell)$, we don't care what happens outside this interval.

Different choices:

① Extend f as a periodic function with period $L = \ell$

<u>example</u>: $f(x) = x^2$, $0 \leq x < 1$

$f(x) =$ 

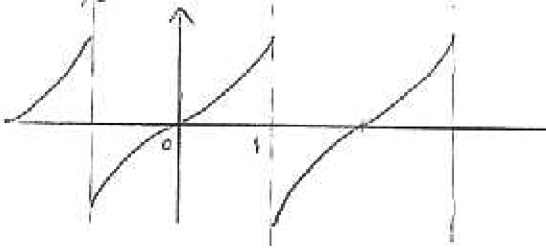
Periodic extension 

The jumps are bad!

② Extend f as a periodic function with period $L = 2\ell$. In this case we get to choose what f looks like over $(-\ell, 0)$.

<u>Example</u>: $f(x) = x^2$, $0 \leq x < 1$

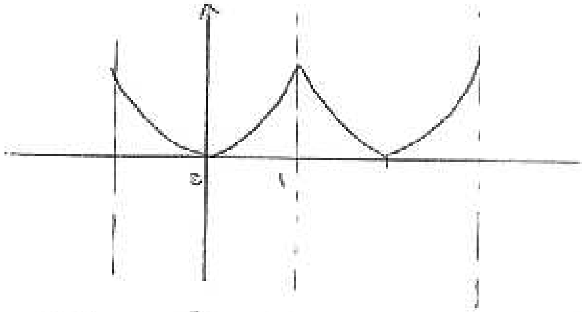
choose $f(x) = -x^2$, $-1 \leq x < 0$

periodic extension $f(x) =$ 

Note: the extended f is an odd function but its jumps are worse than before

example: $f(x) = x^2, \quad 0 \leq x < 1$

choose $f(x) = x^2, \quad -1 \leq x < 0$

periodic extension $f(x) =$ 

Note: the extended f is an even function and the jumps have now disappeared.

<u>Question</u>: What is the best way of extending a function?

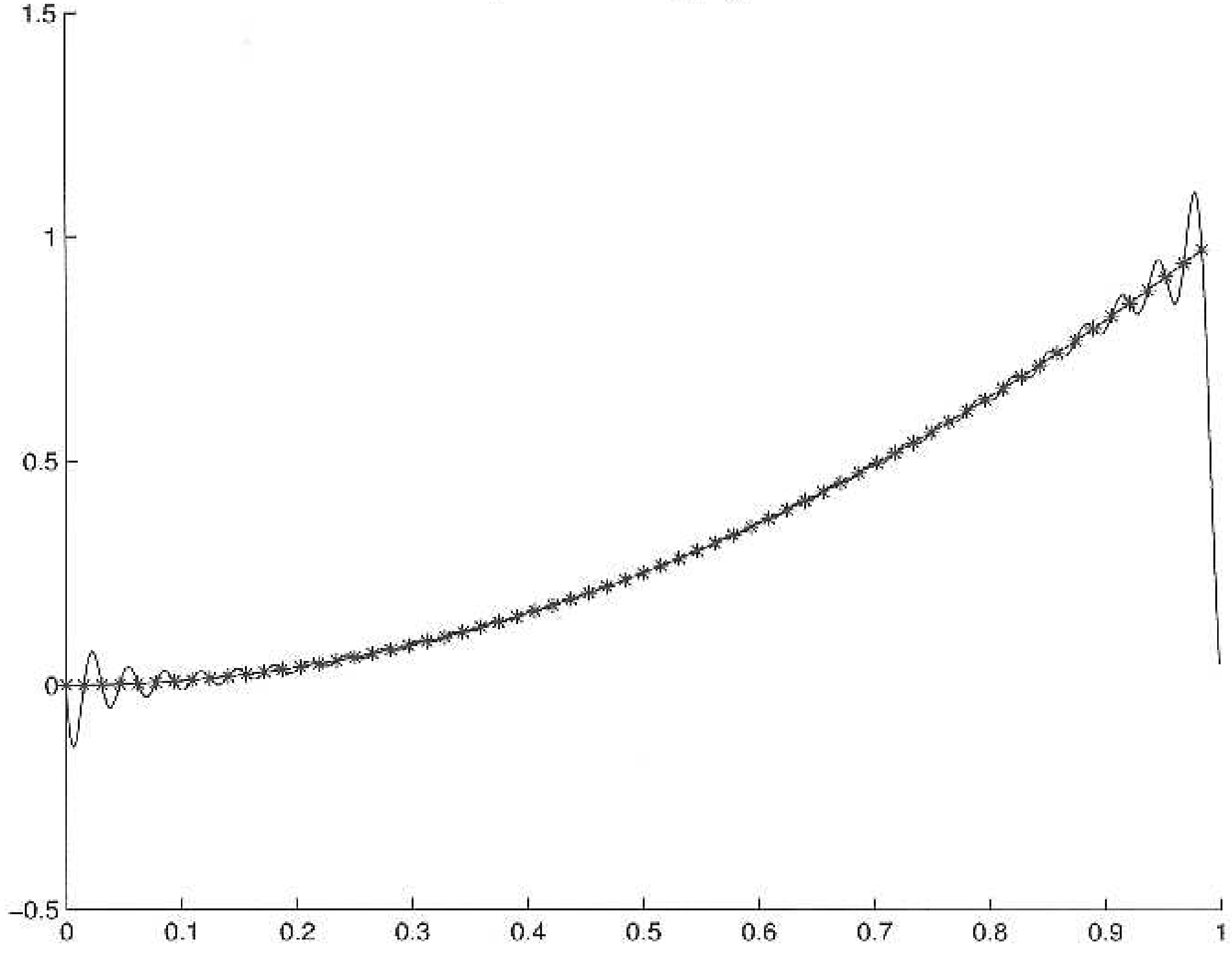
Summary

Given $f(x)$ over $0 \leq x < \ell$, it is possible to define $f(x)$ over $-\ell \leq x < 0$ in such a way that its extended f is either even (cosine series), odd (sine series), or neither even or odd (mixed series). the third choice is seldom useful.

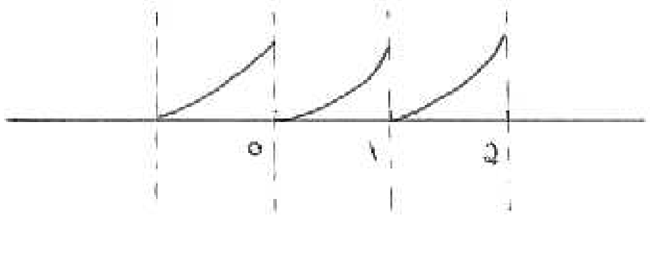
Note: No matter how you extend f, the series always represents the given f over $(0, \ell)$.

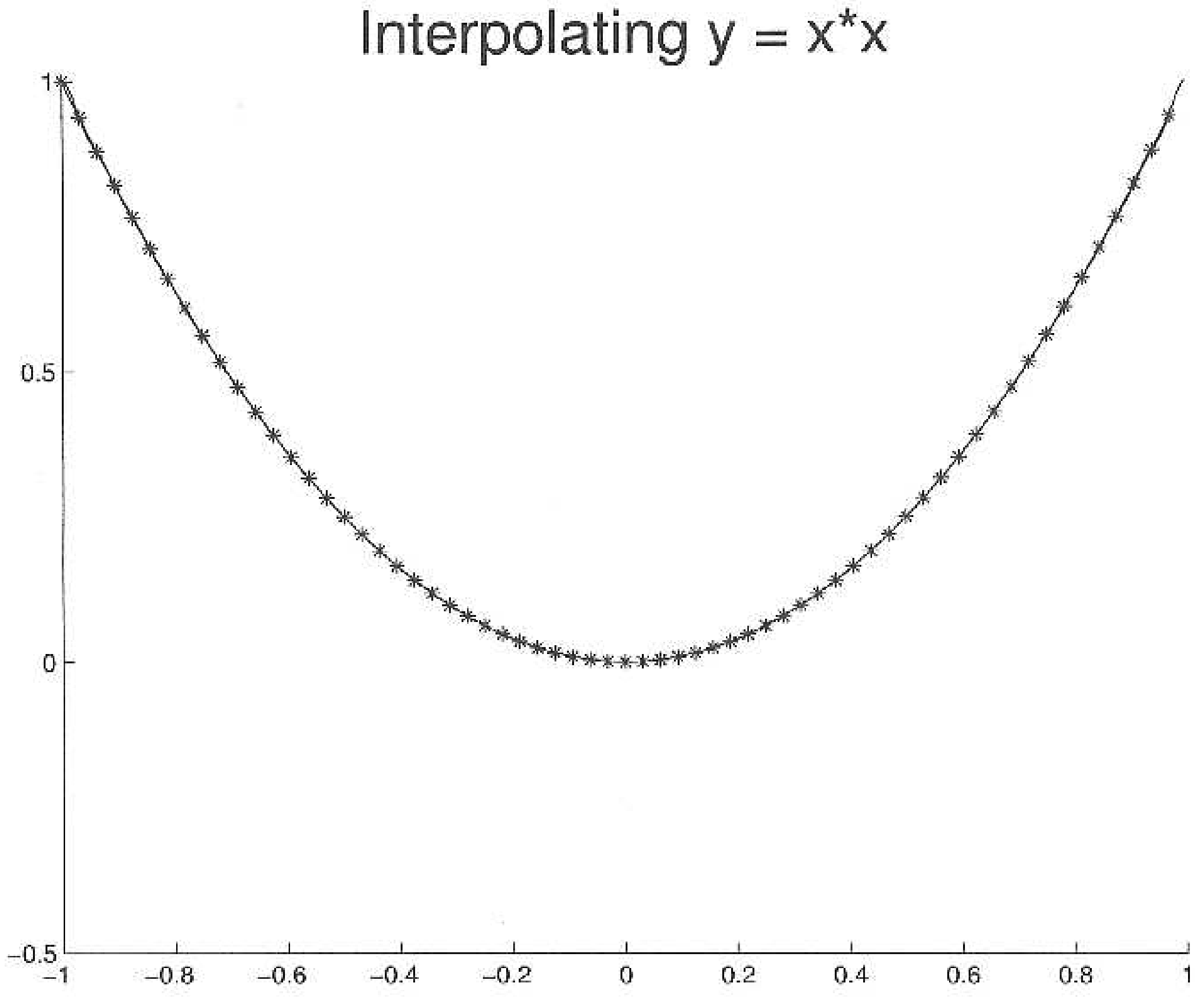
See figures for illustration.

Interpolating y = x*x

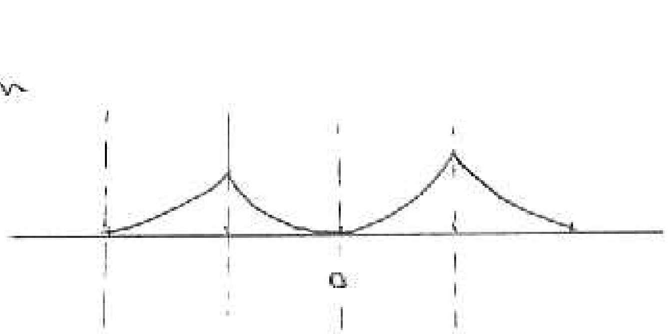


with extension





with extension



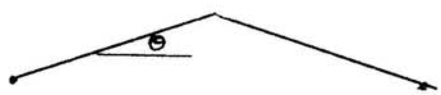
Lecture 5: Wave equation

To do today:

① Derive the wave equation for a vibrating string

② Solve the equation using separation of variables

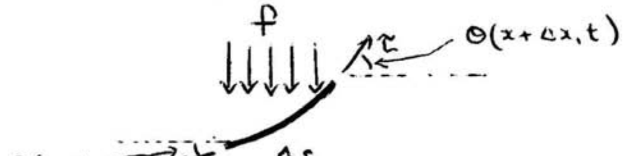
① Derivation: we have a string, fixed at both ends, which is then plucked



Assume: ① deviation is small, $|\theta|<1$

② tension force τ is constant

③ string is flexible (τ acts along string)



σ: mass per unit length

f: body force per unit length

y: deviation of string from normal

Note: $\Delta s \approx \Delta x$ (θ is small)

$$\sin\theta \approx \theta \approx \tan\theta = \frac{dy}{dx} \quad (\theta \text{ is small})$$

Now we use Newton:

$$\sigma \Delta x \frac{d^2y}{dt^2} = \tau \sin\theta(x+\Delta x, t) - \tau \sin\theta(x,t) - f \Delta s$$

use assumptions

$$\approx \tau \Delta x \frac{\partial}{\partial x} \sin\theta(x,t) - f \Delta x$$

where we used $\left\{ \frac{\partial}{\partial x} \sin\theta(x,t) \approx \frac{\sin\theta(x+\Delta x,t) - \sin\theta(x,t)}{\Delta x} \right.$

$$\Rightarrow \quad \sigma \frac{d^2y}{dt^2} \approx \tau \frac{\partial}{\partial x} \sin\theta(x,t) - f$$

or

$$\boxed{\tau \frac{\partial^2 y}{\partial x^2} - \sigma \frac{\partial^2 y}{\partial t^2} = f}$$

- If f is gravitational force : $f = \sigma g$

$$\tau \frac{\partial^2 y}{\partial x^2} - \sigma \frac{\partial^2 y}{\partial t^2} = \sigma g$$

- If σg can be ignored

$$\left(\frac{\tau}{\sigma}\right) y_{xx} - y_{tt} = 0 \qquad y_{xx} = \frac{\partial^2 y}{\partial x^2} \quad y_{tt} = \frac{\partial^2 y}{\partial t^2}$$

or

$$\boxed{c^2 y_{xx} - y_{tt} = 0}$$

Homework: show that c has dimensions of velocity.

Two dimensional version

$$\boxed{c^2(w_{xx} + w_{yy}) - w_{tt} = 0}$$

(2) Solution by Separation of Variables

Consider: $$c^2 y_{xx} - y_{tt} = 0$$

with $y(0,t) = 0 = y(L,t)$ ends of string are fixed

$y(x,0) = f(x)$ initial displacement of string.

$y_t(x,0) = g(x)$ initial velocity of string

Try a solution:

$(*)$ $y(x,t) = X(x)T(t)$ ← this is the crucial idea to this method of solution, rest is all mechanical

then $$c^2 X'' T - X T'' = 0$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

since X and T are independent of each other by $(*)$, this equals a constant, for convenience we choose $-\frac{1}{k^2}$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\frac{1}{k^2}$$

$$\Rightarrow X'' + k^2 X = 0$$

and $$T'' + c^2 k^2 T = 0$$

$$X(x) = \begin{cases} A\cos kx + B\sin kx, & k \neq 0 \\ C + Dx, & k = 0 \end{cases}$$

$$T(t) = \begin{cases} E\cos ckt + F\sin ckt, & k \neq 0 \\ G + Ht, & k = 0. \end{cases}$$

- First investigate $k=0$.

$$y(x,t) = (A+Bx)(F+Gt)$$

Since $y(0,t) = 0 = y(L,t)$ we have $A=B=0$

- For $k \neq 0$

$$y(x,t) = (A\cos kx + B\sin kx)(E\cos ckt + F\sin ckt)$$

$$y(0,t) = 0 \quad \Rightarrow \quad A = 0$$

$$y(L,t) = 0 \quad \Rightarrow \quad \sin kL = 0 \quad \Rightarrow \quad kL = n\pi \quad n = 1, 2, \ldots$$

$$\Rightarrow \quad y(x,t) = B\sin\frac{n\pi}{L}x\left[E\cos\frac{cn\pi}{L}t + F\sin\frac{cn\pi}{L}t\right] \quad n = 1, 2, \ldots$$

$$= \sin\frac{n\pi}{L}x\left[\frac{E}{B}\cos\frac{cn\pi}{L}t + \frac{F}{B}\sin\frac{cn\pi}{L}t\right]$$

Note: we have many solutions, one for each value of n. the most general solution is then

$$y(x,t) = \sum_{n=1}^{\infty}\left(E_n\cos\frac{cn\pi}{L}t + F_n\sin\frac{cn\pi}{L}t\right)\sin\frac{n\pi}{L}x$$

Since $y(x,0) = f(x)$ (5)

we have
$$f(x) = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{L}$$

Note: we need the sine expansion of $f(x)$ (Fourier). Period is $2L$, $f(x)$ is defined over $[0,L]$, i.e. extend in odd manner over $[-L,0]$. Then we have all the F_n's.

Since $y_t(x,0) = g(x)$
$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} E_n \sin \frac{n\pi x}{L}$$

Again we need the sine expansion of $g(x)$ to get the coefficients E_n.

Lecture 6: Dimensional Analysis

The idea: Reduce the number of parameters

Determine which terms are small enough to be ignored.

Simplify the equations.

Demonstrate by means of example:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2}, \quad x(0)=0 \quad \frac{dx}{dt}(0) = V.$$

describes a particle which is radially projected from the surface of the earth

- g - gravitational accelleration
- R - radius of earth
- V - initial velocity
- x - distance above surface of earth.

The magnitudes of these quantities depend on the units that are used: feet, meter, mile, etc. In order to remove this dependence, we need to put the equation in non dimensional form. Use a two-step procedure.

Step A: List all the parameters and variables together with the dimensions

x	-	length
t	-	time
g	-	length * time^{-2}
V	-	length * time^{-1}
R	-	length

Step B: For a variable s. Form a combination of variables p with the <u>same</u> dimension as s. then s/p is a new dimensionless variable.

Back to our example:

x has the same dimension as R. Therefore let $y = x/R$. Also let $\tau = t/\alpha$ where α is yet to be determined but has the same dimension as t.

Then
$$\frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(Ry) = R\frac{d^2y}{dt^2}$$
$$= R\frac{d}{dt}\left(\frac{dy}{d\tau}\cdot\frac{d\tau}{dt}\right) = \frac{R}{\alpha}\frac{d}{dt}\left(\frac{dy}{d\tau}\right)$$
$$= \frac{R}{\alpha^2}\frac{d^2y}{d\tau^2}.$$

We now have

$$\left(\frac{R}{\alpha^2}\right)\frac{d^2y}{d\tau^2} = \frac{-gR^2}{(Ry+R)^2} = \frac{-g}{(1+y)^2}$$

or $$\frac{R}{g\alpha^2}\frac{d^2y}{d\tau^2} = \frac{-1}{(1+y)^2}$$

with $y(0)=0$, $\quad \frac{R}{\alpha}\frac{dy}{d\tau} = V \quad$ or $\quad \frac{dy}{d\tau} = \frac{\alpha V}{R}$

One possibility is to choose $\alpha = \frac{R}{V}$:

I. $$\frac{V^2}{gR}\frac{d^2y}{d\tau^2} = \frac{-1}{(1+y)^2}, \quad y(0)=0, \quad \frac{dy}{d\tau} = 1.$$

Or another possibility is to choose $\alpha^2 = \frac{R}{g}$:

II. $$\frac{d^2y}{d\tau^2} = \frac{-1}{(1+y)^2}, \quad y(0)=0, \quad \frac{dy}{d\tau} = \frac{V}{\sqrt{gR}}$$

Or:

I. $$\varepsilon\frac{d^2y}{d\tau^2} = \frac{-1}{(1+y)^2}, \quad y(0)=0, \quad \frac{dy}{d\tau} = 1$$

II. $$\frac{d^2y}{d\tau^2} = \frac{-1}{(1+y)^2}, \quad y(0)=0, \quad \frac{dy}{d\tau} = \sqrt{\varepsilon}.$$

met $\varepsilon = \frac{V^2}{Rg}$.

Question: If $\varepsilon \ll 1$, can the terms containing ε be ignored?

No! Try it. (Why not?)

Question: What is the physical meaning of $\varepsilon \ll 1$?

Answer: Although $\varepsilon << 1$ we have no idea (yet) whether $\varepsilon \frac{d^2y}{dt^2}$ is small (in comparison with $-\frac{1}{(1+y)^2}$), or whether $\varepsilon^{1/2}$ is small in comparison with $\frac{dy}{d\tau}$.

We have to do a little more work.

We need to choose a specific scaling such that if a term is multiplied with a small dimensionless quantity, we know that the product is small. In our example, $\varepsilon \frac{d^2y}{d\tau^2}$ is only small for $\varepsilon << 1$ if $\frac{d^2y}{d\tau^2}$ is $O(1)$ as $\varepsilon \to 0$.

Now investigate the magnitude of the different terms in the original equation.

If $\varepsilon = V^2/gR << 1$, it means that initial velocity is small, i.e. $x << R$. Thus

$$\frac{d^2x}{dt^2} \approx -g.$$

With initial velocity V and $\frac{d^2x}{dt^2} = -g$, the maximum value of x is $\frac{1}{2} V^2/g$. As an order of magnitude $x \sim V^2/g$.

<u>Note</u>: Returning to ε, for x to remain small, relative to R, we need $\frac{V^2}{g} << R$ (rather that $V^2 << Rg$).

Now we do another scaling. If

$$y = \frac{x}{(V^2/g)}$$

we know that the magnitude of y is of order 1. Let $\tau = \dfrac{t}{\alpha}$:

$$\left(\frac{V^2}{g\alpha^2}\right)\frac{d^2y}{d\tau^2} = \frac{-gR^2}{\left(V^2/g \cdot y + R\right)^2}$$

$$\left(\frac{V}{g\alpha}\right)^2 \frac{d^2y}{d\tau^2} = \frac{-1}{\left(1 + \frac{V^2}{gR}y\right)^2} = \frac{-1}{(1+\varepsilon y)^2}$$

Since we know

$$-g \approx \frac{d^2x}{dt^2} = \left(\frac{V^2}{g\alpha^2}\right)\frac{d^2y}{d\tau^2}$$

choose α such that

$$\frac{V^2}{g\alpha^2} = g \quad \Rightarrow \quad \alpha = \frac{V}{g}.$$

We have then:

$$\frac{d^2y}{d\tau^2} = \frac{-1}{(1+\varepsilon y)^2}, \quad y(0)=0, \quad \frac{dy}{d\tau} = 1.$$

Important Note:

We know that y is of order 1, therefore εy can be ignored in comparison to 1 if $\varepsilon << 1$. Said differently:

If $\varepsilon << 1$ then $\varepsilon y << 1$

and $1 + \varepsilon y \approx 1$.

Thus for $\varepsilon << 1$ we have the approximation

$$\frac{d^2 y}{dt^2} = -1 \quad , \quad y(0) = 0 \quad , \quad \frac{dy}{d\tau} = 1 .$$

This is a very good approximation.

Lecture 7 : d' Alembert's Solution

Objective: Derive and apply d' Alembert's solution of the wave equation

Wave equation : $c^2 y_{xx} = y_{tt}$

Change* of variables :

$$\xi = x - ct \quad , \quad \eta = x + ct$$

* This forms part of a larger body of knowledge, so cannot be motivated fully here

Then $$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$= \frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}$$

but we want the second derivative

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial^2 y}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial x} + \frac{\partial^2 y}{\partial \xi \partial \eta} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial^2 y}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial x}$$

or equivalently,

$$y_{xx} = y_{\xi\xi} + y_{\eta\xi} + y_{\xi\eta} + y_{\eta\eta}$$

Similarly: $\dfrac{\partial y}{\partial t} = y_\xi \dfrac{\partial \xi}{\partial t} + y_\eta \dfrac{\partial \eta}{\partial t} = -c\, y_\xi + c\, y_\eta$ (2)

$$\frac{\partial^2 y}{\partial t^2} = c^2 y_{\xi\xi} - c^2 y_{\eta\xi} - c^2 y_{\xi\eta} + c^2 y_{\eta\eta}$$

The wave equation then becomes:

$$c^2 y_{\xi\xi} + c^2 y_{\eta\xi} + c^2 y_{\xi\eta} + c^2 y_{\eta\eta} = c^2 y_{\xi\xi} - c^2 y_{\eta\xi} - c^2 y_{\xi\eta} + c^2 y_{\eta\eta}$$

$$2c^2 \left[y_{\eta\xi} + y_{\xi\eta} \right] = 0$$

If $y_{\eta\xi} = y_{\xi\eta}$ then

$$\boxed{y_{\xi\eta} = 0}$$

$$\frac{d}{d\eta} y_\xi = 0$$

$$\Rightarrow \quad y_\xi = G'(\xi)$$

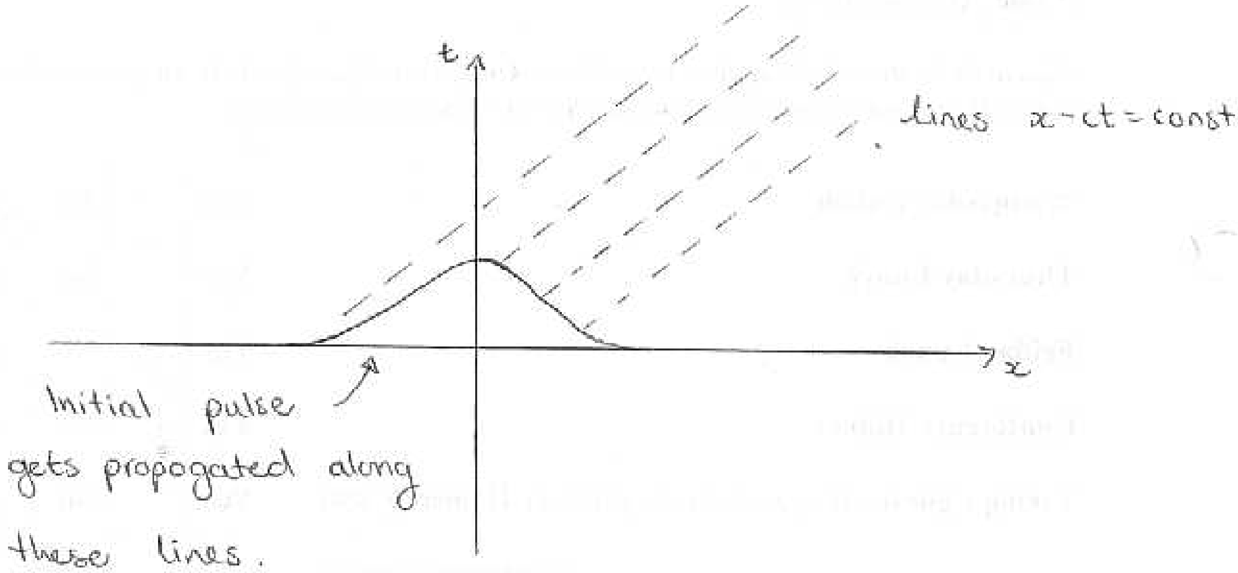
$$y = G(\xi) + F(\eta)$$

$$\boxed{\Rightarrow \quad y(x,t) = G(x - ct) + F(x + ct) \qquad \text{d'Alembert's solution.}}$$

Homework: Write the solution you get from separation of variables in this form above.

Note: $G(x-ct)$ remains constant where

$x - ct = \text{constant}.$



The initial slope of the wave determines the expessions for F and G.

Example 1: Infinite string

$$c^2 y_{xx} = y_{tt}, \quad -\infty < x < \infty, \; -\infty < t < \infty$$

$$y(x,0) = f(x), \quad y_t(x,0) = g(x).$$

We have

$$f(x) = y(x,0) = G(x) + F(x) \qquad ①$$

$$g(x) = y_t(x,0) = -c\,G'(x) + c\,F'(x)$$

We want to solve for F and G in terms of f and g. Two equations and two unknowns. ⇒ simultaneous equations.

Integrate from 0 to x:

$$\int_0^x g(s)\,ds = -c\,G(x) + c\,F(x) + c\,G(0) - c\,F(0) \quad ②$$

multiply ① by c and add $c \times ①$ to ②

$$cf(x) + \int_0^x g(s)\,ds = 2c\,F(x) + c\,G(0) - c\,F(0)$$

$$F(x) = \tfrac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds - \tfrac{1}{2}\left[G(0) - F(0)\right]$$

multiply ① by c and substract ② from $c \times ①$

$$cf(x) - \int_0^x g(s)\,ds = 2c\,G(x) - c\left[G(0) - F(0)\right]$$

$$\Rightarrow \quad G(x) = \tfrac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds + \tfrac{1}{2}\left[G(0) - F(0)\right]$$

Since $\quad y(x,t) = G(x-ct) + F(x+ct)$

$$= \tfrac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds + \tfrac{1}{2}\left[G(0) - F(0)\right]$$

$$+ \tfrac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds - \tfrac{1}{2}\left[G(0) - F(0)\right]$$

$$= \tfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

So here we have a general solution, in terms of any given initial conditions.

Example:

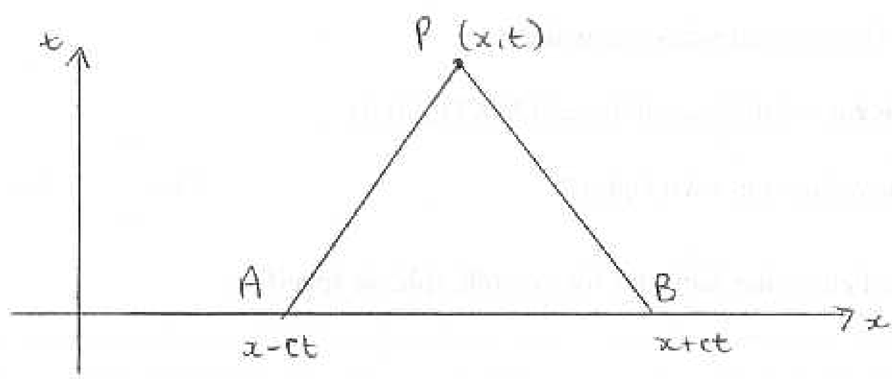
If $\quad y_t(x,0) = g(x) = 0 \qquad y(x,0) = f(x)$

$$y(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right]$$

Note: The initial pulse splits into a right-going and a left-going pulse.

The lines
$$x - ct = \text{constant}$$
$$x + ct = \text{constant}$$
are the characteristics.

Example:



Data $(f(x), g(x))$ on interval AB determines the solution inside triangle ABP.

In case of boundaries (ie we don't have $-\infty < x < \infty$), the waves are reflected off the boundaries.