Question 1. Explain what it means for a function $f\colon \mathbb{R} \to \mathbb{R}$

- to be differentiable at the point $a \in \mathbb{R}$;
- to have a limit at the point $a \in \mathbb{R}$.

*** Question 2 [4+2 marks].**

(1) Show, directly from the definition, that the function given by $f(x) = x^2$ is differentiable at a point a and has derivative $f'(a) = 2a$.

(2) Show, directly from the definition, that the function given by $f(x) = x^n$ for an integer $n > 0$, is differentiable at a point a and has derivative $f'(a) = na^{n-1}$; you may use the identity for $(x^n - a^n)$ from lectures.

(3) Show, using induction on n and any results from lectures you wish, that the function given by $f(x) = x^n$ for an integer $n > 0$, is differentiable at a point a and has derivative $f'(a) = na^{n-1}$; you should explain what results from lectures you are using.

(4) Show, using induction on n and any results from lectures you wish, that the function given by $f(x) = x^n$ for an integer $n < 0$, is differentiable at a point a and has derivative $f'(a) = na^{n-1}$; you should explain what results from lectures you are using.

*** Question 3 [1+2 marks].** If f and g are functions $\mathbb{R} \to \mathbb{R}$ differentiable at $a \in \mathbb{R}$, show that $(f + g)$ is also differentiable at a.

Question 4. Suppose $h\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3 + x^2 + c & \text{if } x \geqslant 0 \end{cases}$$

for some constant $c \in \mathbb{R}$. Show that there is exactly one value of c which makes h differentiable at 0. You should justify your answer, explaining which results (if any) from lectures you are using.

For that value of c, is h', thought of itself as a function $\mathbb{R} \to \mathbb{R}$, differentiable at 0?

* **Question 5 [2+1 marks].** Find all maximum and minimum points of the following functions as x ranges over the sets of numbers given, briefly justifying your answers.

(1) $f(x) = x^3 - 12x + 208$ where $x \in \mathbb{R}$.
(2) For fixed $a > 0$, $f(x) = a^{-x}$ where and $x \in [0, 1]$.
(3) $f(x) = x^2$ where $x \in (0, 1)$.
(4) $f(x) = \cos(x)$ where $x \in [0, 7]$.

* **Question 6 [3+2 marks].** Suppose f and g are differentiable functions $\mathbb{R} \to \mathbb{R}$ and suppose their derivatives, $f'(x)$ and $g'(x)$, are continuous as functions of x. Suppose $f(a) = g(a) = 0$ for some a, but $g'(a) \neq 0$. By considering the equality (for $x \neq a$)

$$\frac{f(x) - f(a)}{x - a} \cdot \frac{x - a}{g(x) - g(a)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

show that $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$. This result is known as *l'Hôpital's rule.*

Hence, or otherwise, compute $\lim_{x \to 0} \left(\frac{\sin(3x)}{2\cos(x) - 2 + x} \right)$.

Using an illustration of the graphs of a simple example of the form $f(x) = bx$ and $g(x) = cx$, give an intuitive, geometric explanation of why l'Hôpital's rule should be true.

* **Question 7 [2+1 marks].** Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at least two times at all points of $\mathbb{R}$. Suppose that for some points $a < b < c$, we have $f(a) = f(b) = f(c)$. Use Rolle's theorem to show there is a point $d \in (a, c)$ with $f''(d) = 0$.

Question 8. Write $f^{(n)}(x)$ for the n^{th} derivative of f at x. Find a formula, in terms of n, for $f^{(n)}(0)$ where f is each of the functions

(1) $\sin(x)$;
(2) $\cos(2x)$;
(3) $\sin^2(x)$.