ed in the basis, so it works
say, "It has been proved
tion step proves that we
to the next one.
an infinite "topple the
a long line of dominos
ed so that each domino
ext one (induction step).
Then it seems that all
s likely that the whole
e collisions on a freeway.
an infinitely tall ladder.
first rung. You use the
ung you have reached to

the mathematical induc-
enth rung is broken or
the ground for you to
feet?

ay of real dominos and
al induction.

ith binary operation $+$. Use
ociative law, $a + (b + c) =$
of the inductive definition of
the semigroup N?

hold up his fingers separated
r a class how associativity of
mmutativity of addition.

with multiplication as binary
ve by induction on k that
old in this case?

f a semigroup with operation
$A \theta A^n = A^n \theta A$. This result
w, for instance, that $A \theta A^2 =$
s, A^3 would be ambiguous if

ed at the outset as assumed
ulates such as those of Peano

Introduction to Abstract Algebra, Wiley, 1940, pp. 1–5; or E. Landau, *Foundations of Analysis*, Chelsea, 1960.)

Countable Infinity

The natural numbers taken as a whole have the peculiar property that they can be put into one-to-one correspondence with a proper subset of themselves. (By a "proper" subset we mean a set that has some but not all members of the original set.) A classic illustration of the peculiarities of arithmetic in such a case is that of the hotel with rooms numbered 1, 2, 3, . . . , one room for each natural number. Suppose every room is full, but another guest arrives. The manager simply gives him room No. 1, whose former occupant moves to No. 2, whose former occupant moves to No. 3, and so on. What would be an impossible problem in a finite hotel is solved easily in a *countably infinite* hotel.

Definition 2–10. A set is called **countably infinite** if its members can be put into one-to-one correspondence with the natural numbers. The set is then said to have **cardinal number** $\aleph_0$ (read "a'-leph null," from the first letter of the Hebrew alphabet). A finite set of n members has **cardinal number n** and is also said to be countable.

2. The integers I. The natural numbers N form a semigroup under $+$ (see Exercise 2–4) but not a group. The integers I, 0, 1, -1, 2, -2, 3, . . . , represent the completion of N under subtraction; that is, they include the additive identity and inverses, so that subtraction, the inverse of addition, becomes a binary operation on I.

If 0 is to serve as the additive identity, it has to be the solution for $\aleph_0$ problems:

$$1 + x = 1, \quad 2 + x = 2, \quad 3 + x = 3, \ldots,$$

(see Definition 1–2 *ii*). One way of adjoining an additive identity to N is to lump all these problems together in an "equivalence class" and equate the answer to the problems: Dropping all but the pair of numbers (a, b) in the problem $a = b + x$, we write the equivalence class as

$$\{(1, 1), (2, 2), (3, 3), \ldots\}$$

and call it 0 for short.