It is possible to build structures which combine groups and rings. Given a ring R and group G, we define $R[G]$ to be the set of all finite formal sums $\{\sum r_i g_i \mid r_i \in R,\ g_i \in G\}$. As with polynomials we add elements component-wise and multiply as if the distributive law holds and elements of R commute with those of G:

$$(r_1 g_1 + r_2 g_2 + \cdots + r_n g_n)(s_1 h_1 + \cdots + s_m h_m) = \sum_{i,j} r_i s_j g_i h_j.$$

This makes $R[G]$ into a ring, called a *group ring*. The multiplicative identity is $1e$, where $1, e$ are the identities of R and G respectively. The additive identity is any sum with all $r_i = 0$, so just write it as 0. [redacted]

a. Let $R = \mathbb{Z}/2\mathbb{Z}$ and $G = \mathbb{Z}_2$. Find a quotient of a polynomial ring which is isomorphic to $R[G]$.

b. For the remainder of the problem, let $R = \mathbb{Z}$ and let G be a finite group in which every nonidentity element has order 2 (called an elementary 2-group). Show that $\mathbb{Z}[G]$ has zero divisors.

c. Show that there is a one-to-one correspondence between ring homomorphisms $\psi\colon \mathbb{Z}[G] \to \mathbb{Z}$ and group homomorphisms $\chi\colon G \to \{\pm 1\}$.

d. The augmentation mapping is the homomorphism $\mathbb{Z}[G] \to \mathbb{Z}$ defined by sending all $g \in G$ to 1. Let I be the kernel of this homomorphism. What are all the maximal ideals containing I?

e. Let P be a minimal prime ideal of $\mathbb{Z}[G]$. You may assume that $P \cap \mathbb{Z} = \{0\}$. Let M be the maximal ideal from (d) that contains I and 2. Show that $P \subseteq M$.