UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 2
2003/2004

MAS365

Queueing systems

Time allowed: 1 hour 30 minutes

Credit will be given for ALL answers to questions in Section A, and for the best TWO answers to questions in Section B. No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.

Marks allocated to each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are FIVE questions in Section A and THREE questions in Section B.

Statistical tables will be provided.

SECTION A

A1. A Poisson process of rate 2 per hour models births in the maternity ward of a hosptial. Let $N(t)$ stand for the number of births in the first t hours and X the time of the first birth in hours.

(a) What are the probability distributions of $N(0.25)$, $N(0.75)$, $N(1)$ and X?

(b) What is the probability that at least 2 babies are born in the first 15 minutes?

(c) Given that exactly 4 babies were born in the first hour, what is the probability that they were all born in the first 15 minutes?

[10 marks]

A2. Customers arrive at a service facility modelled as an $M/M/1$ queue, which is in steady-state. Arrivals occur according to a Poisson process with rate 12 per hour. Service times per customer are exponentially distributed with mean 4 minutes.

(a) A steady-state exists for this system. Explain and justify this statement.

(b) What is the probability that an arriving customer will be serviced immediately upon arrival at the facility?

(c) What is the proportion of time that the service facility is busy?

(d) What is the expected number of customers in the queue?

[6 marks]

A3. Consider the following probability density function, f, of the random variable X

$$f(x) = \begin{cases} 0, & x < -2 \\ \frac{1}{4}(x+2), & -2 \le x \le 0 \\ \frac{e^{-x}}{2}, & x > 0. \end{cases}$$

(a) Compute and sketch the distribution function F of the random variable X.

(b) Given that 0.132 and 0.923 are two realised values of $U \sim U(0,1)$, obtain two corresponding realised values of X.

[12 marks]

A4. Consider an $M/M/1$ queueing system with fixed arrival rate λ, but where we may choose the service rate μ. We want to choose the value of $\mu > \lambda$ that will minimise the total cost of waiting and serving per unit time in the steady-state

$$TC(\mu) = C_1\mu + C_2L_s$$

where

$$C_1 = \text{cost per unit increase in } \mu \text{ per unit time}$$
$$C_2 = \text{cost of waiting per unit waiting time per customer}$$

and L_s is the mean number of customers in the system.

(a) Show that the optimum value of μ is

$$\hat{\mu} = \lambda + \sqrt{\frac{C_2\lambda}{C_1}}.$$

(b) Jobs arrive at a machine shop according to a Poisson process at the rate of 10 per day. A single machine is available for use. It is estimated that a unit increase in the production rate of the machine will cost £100 per week. Delayed jobs normally result in lost business, which is estimated to be £200 per job per week. Determine the optimum speed of the machine in units of the production rate.

[8 marks]

A5. A service station has 5 service points each of which can work at rate μ per hour. Arrivals at the system form a Poisson process with rate λ per hour, but there is only space in the system for 8 customers and arrivals finding a full system are lost.

Formulate this as a Birth-Death model. Does a steady-state exist?

[4 marks]

SECTION B

B6. A birth-death model has arrival rates $\{\lambda_n, n \geq 0\}$ and service rates $\{\mu_n, n \geq 1\}$ where λ_n and μ_n are respectively the arrival rate and service rate when there are n customers in the system. We write $p_n(t)$ for the probability of n customers in the system at time $t \geq 0$ with p_n for the corresponding steady-state probability.

(a) Show that the following equations are satisfied:

$$\begin{aligned}\dot{p}_0(t) &= -\lambda_0 p_0(t) + \mu_1 p_1(t)\\ \dot{p}_n(t) &= -(\lambda_n + \mu_n)p_n(t) + \mu_{n+1}p_{n+1}(t) + \lambda_{n-1}p_{n-1}(t), \ \ n \geq 1\end{aligned}$$

and write down the corresponding equations satisifed by the steady-state probabilities $\{p_n, \ n \geq 0\}$.

You may assume that, when a steady-state exists, p_n is given by the formula

$$p_n = \frac{\lambda_0\lambda_1 \ldots \lambda_{n-1}}{\mu_1\mu_2 \ldots \mu_n} p_0, \ \ n \geq 1.$$

(b) Consider the following situation: N machines are working, but each breaks down at constant rate λ, i.e.,

$$\begin{aligned}&P(\text{machine } j \text{ breaks down in } [t, t+h) \text{ when working at } t)\\ &\qquad = \lambda h + o(h), \ \ j = 1, \ldots, n.\end{aligned}$$

A single repairman works at rate μ, i.e.,

$$\begin{aligned}&P(\text{repair completed in } [t, t+h) \text{ when repairman busy at } t)\\ &\qquad = \mu h + o(h).\end{aligned}$$

This situation can be thought of as a queueing system with broken machines "arriving" in the queue awaiting service by the repairman.

(i) Explain why this system can be thought of as a birth-death model with

$$\begin{aligned}\lambda_n &= \begin{cases}\lambda(N-n), & n = 0, 1, \ldots, N\\ 0, & n \geq N+1,\end{cases}\\ \mu_n &= \mu, \text{ for all } n \geq 1.\end{aligned}$$

Question B6 continued on next page

(ii) Explain why a steady-state exists for all λ, μ.

(iii) Show that

$$p_n = \frac{\frac{N!}{(N-n)!}\rho^n}{F(N,\rho)}, n = 0, 1, \ldots, N,$$

where

$$F(N,\rho) = \frac{1}{p_0} = \sum_{n=0}^{N} \frac{N!}{(N-n)!}\rho^n$$

and $\rho = \frac{\lambda}{\mu}$.

Two key quantities are:

- *Operator utilisation (P)*: the proportion of time the repairman is working;
- *Machine efficiency (E)*: the ratio of the total production achieved to that which would have been acheived had no stoppages taken place.

It can be shown that

$$E = \frac{P}{N\rho}.$$

(iv) Deduce that

$$E = \frac{F(N-1,\rho)}{F(N,\rho)}.$$

[30 marks]

B7. Consider an $M/M/1$ queue in which arrival rates are modelled by a Poisson process of rate λ and service times are exponentially distributed with mean $\frac{1}{\mu}$. As usual we write $\rho = \lambda/\mu$, $p_n(t)$ for the probability of n customers in the system at time $t \geq 0$, and p_n for the corresponding steady-state probability. The following difference-differential equations can be obtained:

$$\begin{aligned} \dot{p}_0(t) &= -\lambda p_0(t) + \mu p_1(t), \\ \dot{p}_n(t) &= -(\lambda + \mu) p_n(t) + \mu p_{n+1}(t) + \lambda p_{n-1}(t), \ n \geq 1. \end{aligned}$$

(a) Write down the balance equations satisfied by the steady-state probabilities $\{p_n, n \geq 0\}$.

(b) The steady-state probabilities p_n of there being exactly n customers in the system can be written as:

$$p_n = \rho^n p_0, \ n = 0, 1, 2, \ldots$$

Deduce that $p_0 = 1 - \rho$.

(c) Show that L_s, the mean number of customers in the system, can be written as

$$L_s = \frac{\rho}{1 - \rho}$$

and obtain a formula for L_q, the expected number of customers in the queue, W_s, the expected waiting time in the system and W_q, the expected waiting time in the queue.

(d) A fast-food restaurant has *one* drive-in window. It is estimated that cars arrive according to a Poisson process with rate 2 every 5 minutes and that there is enough space to accommodate a line of 10 cars. Other arriving cars can wait outside this space, if necessary. It takes 1.5 minutes on average to complete an order, and the service times vary according to an exponential distribution. Determine the following:

(i) The probability that the facility is idle.

(ii) The expected number of customers waiting but currently not being served.

(iii) The expected waiting time until a customer can place his order at the window.

Question B7 continued on next page

(iv) The probability that the waiting line will exceed the capacity of the space leading to the drive-in window.

(v) Determine the probability that the waiting time per customer will exceed the average waiting time in the queue assuming that a FCFS (First Come First Served) service discipline is in operation.

(vi) To attract more business, the owner of the fast-food restaurant decides to give a free drink to each customer who waits more than 5 minutes for service. Normally, a drink costs 50 pence. How much is the owner expected to pay daily for free drinks? Assume that the restaurant is open for 12 hours daily.

[30 marks]

B8. In an $M/G/1$ queueing system, customers arrive according to a Poisson process (rate λ) and are served by a single server. Service times are i.i.d. and have probability density function, $f(t),\ t \geq 0$.

Label the customers $1, 2, \ldots, n, \ldots,$ in order of their leaving the system and write

q_n = the number of customers left in the system just after the n^{th} customer has left.

A_n = the number of customers who arrive at the system during the service time of customer n.

T = a general service time (with p.d.f. $f(t)$).

(a) Prove that $A_1, A_2, \ldots,$ are i.i.d. random variables with probability distribution given by

$$P(A_n = k) = \frac{\lambda^k}{k!}\int_0^\infty t^k e^{-\lambda t} f(t)dt$$

and that $E[A_n] = \lambda E[T]$.

(b) The Pollaczek-Khintchine formula for the expected number of customers in an $M/G/1$ queueing system is given by

$$L_s = \lambda E(T) + \frac{\lambda^2 \left\{(E[T])^2 + Var(T)\right\}}{2\{1 - \lambda E[T]\}}.$$

An $M/G/1$ queue has a service time with a gamma distribution with parameters n and α, where n is a positive integer. Thus the density function, $f(t)$, is given by

$$f(t) = \frac{\alpha^n t^{n-1} e^{-\alpha t}}{(n-1)!},\ t > 0.$$

(i) Show that $E[T] = \frac{n}{\alpha}$.

(ii) Show, assuming that $\rho = \frac{\lambda n}{\alpha}$ and $Var(T) = \frac{n}{\alpha^2}$, that the expected number of customers in the system is given by

$$\frac{\rho\,[2n - (n-1)\rho]}{2n(1-\rho)}.$$

Question B8 continued on next page

(iii) A new system is being considered for which the customer's service time is a constant with value $\frac{n}{\alpha}$. How will the expected number of customers in the system compare with that of the original system?

[30 marks]