Show that every countable linear ordering is isomorphic to a subset of the rationals under their usual order but that ω_1 (with its well order) is not isomorphic to any set of reals under their usual ordering. The solution may use any algebraic facts about the reals.

Notation and Definitions

ω_1 is an uncountable set

$\leq$ is a well ordering of ω_1 with the property that $\mathrm{seg}_{\omega_1}(\alpha)$ is countable for all $\alpha \in \omega_1$

[from Notes on Set Theory -Yiannis Moschovakis]
[Let me know if you have a question on notation; I have a pdf file of this book]

5.19. Definition. A binary relation $\leq$ on a set P is a **partial ordering** if it is reflexive, transitive and antisymmetric, i.e., for all $x, y, z \in P$,

$$x \leq x \qquad \text{(reflexivity)},$$
$$x \leq y \,\&\, y \leq z \Longrightarrow x \leq z \qquad \text{(transitivity)},$$
$$x \leq y \,\&\, y \leq x \Longrightarrow x = y, \qquad \text{(antisymmetry)}.$$

In connection with partial orderings we will also use the notation

$$x < y \iff_{\mathrm{df}} x \leq y \,\&\, x \neq y.$$

The partial ordering $\leq$ is **total**, or **linear**, or simply an **ordering**, if, in addition, any two elements of P are **comparable** in $\leq$, i.e.,

$$(\forall x, y \in P)[x \leq y \vee y \leq x],$$

or equivalently

$$(\forall x, y \in P)[x < y \vee x = y \vee y < x].$$