UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 1
2003/2004

MAS349

Topology

Time allowed: 1 hour 30 minutes

Credit will be given for ALL answers to questions in Section A, and for the best TWO answers to questions in Section B. No credit will be given for other answers and students are strongly advised not to spend time producing answers for which they will receive no credit.

Marks allocated to each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are THREE questions in Section A and THREE questions in Section B.

SECTION A

A1. (a) State the Axiom of Choice.

(b) Define a *well ordering*. Can every set be well ordered?

(c) Give a well ordering for the rationals. Is the Axiom of Choice needed for this?

[10 marks]

A2. (a) Let X be a nonempty set. Define what is meant by a *topology* τ on X.

(b) Define a *neighbourhood* of a point x in (X, τ).

(c) State what it means for a function $f : X \to Y$, where X and Y are topological spaces, to be *continuous*.

(d) Prove that $f : X \to Y$ is continuous if and only if for each open set H in Y, $f^{-1}(H)$ is open in X.

[15 marks]

A3. (a) Define what it means for a topological space X to be a *Hausdorff space* (ie, a T_2 space).

(b) Define the *product topology* on a product of topological spaces $X = \Pi_\alpha X_\alpha$. Prove that if X is nonempty and each X_α is Hausdorff, then X is Hausdorff.

(c) Define what it means for a topological space to be *compact*. Prove that every closed subset of a compact space is compact.

[15 marks]

SECTION B

B4. (a) Define a *metric space*. What does it mean for two metrics to be *equivalent*? What does it mean for a topological space to be *metrizable*?

(b) Suppose (X, ρ) is a metric space. Define $\tilde{\rho} : X \times X \to \mathbb{R}$ by $\tilde{\rho}(x, y) = \min\{\rho(x, y), 1\}$. Show that $\tilde{\rho}$ is a metric for X.

(c) For the metric space (X, ρ), let $U_\rho(x, r)$ be the open unit disk about the point x of radius r. Prove the following statement: Two metrics ρ and μ for a set X are equivalent if and only if for each $x \in X$ and each $r > 0$, there exist $s, t > 0$ such that $U_\rho(x, s) \subset U_\mu(x, r)$ and $U_\mu(x, t) \subset U_\rho(x, r)$.

(d) Use (c) to show that the two metrics in (b) are equivalent.

[30 marks]

B5. (a) Define a *homeomorphism* between topological spaces X and Y. Define what is meant by a *topological invariant*.

(b) State what it means for a map $f : X \to Y$ to be *open*. Show that a continuous open bijection is a homeomorphism.

(c) (i) Recall that Fr E, the *frontier* of a subset E of a topological space X is defined as $\overline{E} \cap \overline{(X - E)}$. Prove that $\overline{E} = E \cup \text{Fr } E$.

(ii) A topological space is 0-*dimensional* if and only if whenever $x \in V$ and V is open, there is an open set U with empty frontier such that $x \in U \subset V$. Show that the rationals in the relative topology as a subset of $\mathbb{R}$ with the usual topology is a 0-dimensional set.

(iii) Show that being 0-dimensional is a topological invariant.

[30 marks]

B6. (a) Define what it means for a topological space to be *connected.*

(b) Suppose that A and B are subspaces of a topological space X, and that $U \subset A \cap B$ is open in both A and B in the relative topologies. Show that U is open in $A \cup B$ in the relative topology.

(c) Suppose that X is connected and that A is a connected subset of X. Suppose further that $X - A = U \cup V$, where U and V are nonempty disjoint open subsets of $X - A$ in the relative topology for $X - A$. Show that $A \cup U$ is connected.

(d) (i) Define what is meant by a *component* of a topological space.

(ii) A space X is *totally disconnected* if and only if each of its components consists of a point. Show that $\mathbb{Q}$ in the relative topology as a subset of $\mathbb{R}$ with the usual topology is totally disconnected.

(iii) A connected space X is said to have an *explosion point* p if and only if $X - \{p\}$ is totally disconnected. Find an example of such a space.

[30 marks]