The Navier-Stokes equation for a viscous incompressible fluid is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \nu \nabla^2 \mathbf{u},$$

where $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure and ν the kinematic viscosity.

1. Consider inviscid, incompressible, irrotational flow parallel to the xy-plane, with no variation in the z-direction. Explain briefly why the velocity field can be written as the gradient of a potential $\phi(x, y)$, where

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0\,.$$

Verify that the potential

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - 2Ua\theta\,,$$

where $x = r\cos\theta$, $y = r\sin\theta$, can represent such a flow around a solid cylinder $x^2 + y^2 = a^2$. Obtain an expression for the tangential fluid velocity component at the surface of the cylinder.

Use Bernoulli's equation to show that the pressure field on the cylinder is given by

$$p(a, \theta) = C - 2\rho U^2\left(\sin^2\theta + 2\sin\theta\right)\,,$$

where C is a constant and ρ is the fluid density. Hence show that the force exerted by the fluid on the cylinder is $4\pi\rho a U^2$ in the y-direction.

Give a brief qualitative discussion of the relevance of the above model to the physics of aircraft flight.