Ideal Gas Law Derivation

Molecules of the ideal gas are hard spheres that have perfectly elastic collisions with each other and the walls of a cylindrical container with a circular cross sectional area (A) at the end of the cylinder. We will only consider the collisions with the circular cross sectional area (A), which you can think of as the area of a moveable piston at the top of a cylinder. (You will be doing an exercise with such a system in the tutorial on thermodynamics at the end of this lesson). In addition, we have all the molecules moving at the same velocity (v). What follows is a look at the momentum in an elastic collision only in the direction the piston can move.

PART I: In an elastic collision, we know that momentum is conserved (from Module 3). This means that in a collision (mv) before = (mv) after, and in our case the velocity after the collision (v_{after}) is equal to but in the opposite direction of velocity before the collision (v_{before}). This is the case since the molecules moving toward the piston hit it and reverses direction in an elastic collision. Therefore, the change in momentum (Δp) is:

$$\Delta p = mv_{after} - mv_{before};\ \text{but we know that } v_{after} = -v_{before}$$

$$\Delta p = m(-v_{before}) - mv_{before} = -mv_{before} - mv_{before} - 2mv_{before}$$

Since the velocity before = velocity after the change in momentum
Δp = -2mv

PART II: Next we look at mass density (D) of the gas, which is mass per unit volume (Module 4). Since we have the area of the piston (A) and the distance the molecules travel (L), and we know the formula for the volume of a cylinder: Volume = Area of the base x height, we have that in our case Volume = A x L.

$$D = \frac{m}{Vol} = \frac{m}{Area \otimes L} = \frac{m}{AL}$$

Multiply both sides of the equation by AL and we have:

$$DAL = \frac{m}{\cancel{AL}} \otimes \cancel{AL}$$

$$m = DAL$$

Now let's look at the time (Δt) it takes for a molecule traveling at velocity (v) to make a round trip over a distance (d).

We know $v\Delta t = d$ (Module 1) and (d) is the distance of the round trip = 2L we have:

$$v\Delta t = d = 2L$$

$$\frac{v\Delta t}{2} = L;\ \text{substitute for L in the equation } m = DAL:$$

$$m = DA\left(\frac{v\Delta t}{2}\right) = \frac{DAv\Delta t}{2}$$

PART III: Newton's Second Law in alternate form (Module 3) states:

$$\vec{F} = \frac{\text{change in momentum}}{\text{change in time}} = \frac{\Delta p}{\Delta t};\ \text{substitute for } \Delta\text{p} = \text{-2mv (from PART I):}$$

$$\vec{F}_{gas \text{ on } piston} = \frac{\Delta p}{\Delta t} = \frac{-2\Delta(mv)}{\Delta t} = \frac{-2mv}{\Delta t};\ \text{substitute for } m = \frac{DAv\Delta t}{2}\ \text{(from PART II):}$$

$$\vec{F}_{gas \text{ on } piston} = \frac{-2(\frac{DAv\Delta t}{2})v}{\Delta t} = \frac{-DAv^2\Delta t}{\Delta t} = -DAv^2$$

PART IV: Newton's Third Law states:

$$F_{\text{gas on piston}} = -F_{\text{piston on gas}}$$

$$-(-DAv^2) = DAv^2$$

PART V: Pressure is given by

$$P = \frac{F}{A} = \frac{DAv^2}{A} = Dv^2;$$

$$P = Dv^2$$