Show that if M_1 and M_2 are irreducible R modules, then any nonzero R-module homomorphism from M_1 to M_2 is an isomorphism. Deduce that if M is irreducible then $\text{End}_R(M)$ is a division ring (this result is called *Schur's Lemma*). [Consider the kernel and the image.]

Let $\varphi : M_1 \to M_2$ be a nonzero R-module homomorphism. Then $\ker\varphi \neq M$ and $\text{Im}\varphi \neq 0$. But $\ker\varphi$ and $\text{Im}\varphi$ are submodules of M_1 and M_2 respectively, so irreducibility implies $\ker\varphi = 0$ and $\text{Im}\varphi = M_2$, and thus φ is an isomorphism.

If M is irreducible then the above implies that any $\varphi \neq 0$ in the ring $\text{End}_R(M, M)$ has a multiplicative inverse, so $\text{End}_R(M, M)$ is a division ring. ∎

Please clarify the following:

1. Why does it follow from 'Let $\varphi : M_1 \to M_2$ be a nonzero R-module homomorphism.' that then $\ker\varphi \neq M$ and $\text{Im}\varphi \neq 0$.
2. Why does irreducibility imply $\ker\varphi = 0$ and $\text{Im}\varphi = M_2$, and thus φ is an isomorphism.
3. If M is irreducible then why does the above imply that $\varphi \neq 0$ in the ring $\text{End}_R(M, M)$ has a multiplicative inverse, so $\text{End}_R(M, M)$ is a division ring.