Use a u-substitution to evaluate the following integral:

$$\int_{\pi/12}^{\pi/6} \frac{1}{\tan(2\theta)\cdot\cos^2(2\theta)}\,d\theta$$

Solution:

We want to try to find a u-substitution that will make it possible for us to evaluate the resulting u-integral.

Let's take a look at the denominator of the integrand:

$$\tan(2\theta)\cdot\cos^2(2\theta)$$

At first glance, we might think of using $u = \cos(2\theta)$, which would give

$$du = [-\sin(2\theta)]\cdot(2\,d\theta) = -2\sin(2\theta)\,d\theta$$

Therefore,

$$d\theta = \frac{du}{-2\sin(2\theta)}$$

We have to express $-2\sin(2\theta)$ in terms of u. Well, we know that

$$\sin^2(2\theta)+\cos^2(2\theta) = 1,$$

hence

$$\sin(2\theta) = \pm\sqrt{1-\cos^2(2\theta)}$$

Now the interval over which we're integrating (in terms of θ) is

$$\left[\frac{\pi}{12},\frac{\pi}{6}\right],$$

hence

$$\frac{\pi}{6} \le 2\theta \le \frac{\pi}{3}$$

Since

$$0 < \frac{\pi}{6} < \frac{\pi}{3} < \frac{\pi}{2},$$

we see that 2θ lies in the first quadrant for every angle θ in that interval. Therefore, $\sin(2\theta)$ is positive for every angle θ in that interval, hence we use the positive square root:

$$\sin(2\theta) = \sqrt{1-\cos^2(2\theta)}$$

By definition, $\cos(2\theta) = u$, so

$$\sin(2\theta) = \sqrt{1-u^2}$$

Thus

$$-2\sin(2\theta) = -2\sqrt{1-u^2},$$

so

$$d\theta = \frac{du}{-2\sqrt{1-u^2}}$$

We still have to express $\tan(2\theta)$ in terms of u. Using the fact that

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)},$$

we find that

$$\tan(2\theta) = \frac{\sqrt{1-u^2}}{u}$$

Another thing we need to do is convert the θ-limits of integration to u-limits of integration. Since $u = \cos(2\theta)$, we get the following:

$$\begin{aligned} \theta = \frac{\pi}{12} \quad \Longrightarrow \quad u &= \cos\left(2 \cdot \frac{\pi}{12}\right) \\ &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \theta = \frac{\pi}{6} \quad \Longrightarrow \quad u &= \cos\left(2 \cdot \frac{\pi}{6}\right) \\ &= \cos\left(\frac{\pi}{3}\right) \\ &= \frac{1}{2} \end{aligned}$$

Converting everything in the given θ-integral from θ to u, we obtain

$$\begin{aligned} \int_{\pi/12}^{\pi/6} \frac{1}{\tan(2\theta) \cdot \cos^2(2\theta)}\, d\theta \quad &= \quad \int_{\sqrt{3}/2}^{1/2} \frac{1}{\left[\frac{\sqrt{1-u^2}}{u}\right] \cdot u^2} \left[\frac{du}{-2\sqrt{1-u^2}}\right] \\ &= \quad \int_{\sqrt{3}/2}^{1/2} \left(-\frac{1}{2}\right) \frac{du}{u(1-u^2)} \end{aligned}$$

After all that work, we end up having to integrate the function

$$\left(-\frac{1}{2}\right) \frac{1}{u(1-u^2)},$$

which isn't something whose antiderivative we could immediately write down.

However, it turns out that there's a u-substitution that would make it easier to evaluate the original integral.

Our original integrand is

$$\frac{1}{\tan(2\theta)\cdot\cos^2(2\theta)}$$

Recall that

$$\frac{1}{\cos(2\theta)} = \sec(2\theta),$$

so

$$\frac{1}{\cos^2(2\theta)} = \sec^2(2\theta)$$

and we can write the original integral as

$$\int_{\pi/12}^{\pi/6} \frac{\sec^2(2\theta)}{\tan(2\theta)}\,d\theta$$

Now let's let $u = \tan(2\theta)$. Then

$$du = \frac{d}{d\theta}(\tan(2\theta))\,d\theta$$

Recall that

$$\frac{d}{d\theta}(\tan(2\theta)) = [\sec^2(2\theta)]\cdot 2 = 2\sec^2(2\theta),$$

so

$$du = 2\sec^2(2\theta)\,d\theta$$

and

$$\sec^2(2\theta)\,d\theta = \frac{du}{2}$$

Now let's convert our θ-limits of integration to u-limits of integration.

Since $u = \tan(2\theta)$, we get the following:

$$\begin{aligned}\theta = \frac{\pi}{12} \quad\Longrightarrow\quad u &= \tan\left(2\cdot\frac{\pi}{12}\right)\\ &= \tan\left(\frac{\pi}{6}\right)\\ &= \frac{\sin(\pi/6)}{\cos(\pi/6)}\\ &= \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

$$\theta = \frac{\pi}{6} \quad \Longrightarrow \quad u = \tan\left(2 \cdot \frac{\pi}{6}\right)$$
$$= \tan\left(\frac{\pi}{3}\right)$$
$$= \frac{\sin(\pi/3)}{\cos(\pi/3)}$$
$$= \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}$$
$$= \sqrt{3}$$

Converting everything in the given θ-integral to a u-integral, we have

$$\begin{aligned}
\int_{\pi/12}^{\pi/6} \frac{1}{\tan(2\theta)\cdot\cos^2(2\theta)}\,d\theta &= \int_{\pi/12}^{\pi/6} \frac{\sec^2(2\theta)}{\tan(2\theta)} \cdot d\theta \\
&= \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{u} \cdot \frac{du}{2} \\
&= \left(\frac{1}{2}\right) [\ln|u|]_{1\sqrt{3}}^{\sqrt{3}} \\
&= \left(\frac{1}{2}\right) \left[\ln(\sqrt{3}) - \ln(1/\sqrt{3})\right]
\end{aligned}$$

Using the rule of logarithms which states that

$$\ln(a) - \ln(b) = \ln(a/b),$$

and setting a to $\sqrt{3}$ and b to $1/\sqrt{3}$, we obtain

$$\int_{\pi/12}^{\pi/6} \frac{1}{\tan(2\theta)\cdot\cos^2(2\theta)}\,d\theta = \left(\frac{1}{2}\right) \ln\left[\frac{\sqrt{3}}{\left(\frac{1}{\sqrt{3}}\right)}\right]$$

To simplify the fraction

$$\frac{\sqrt{3}}{\left(\frac{1}{\sqrt{3}}\right)},$$

we multiply the numerator ($\sqrt{3}$) by the reciprocal of the denominator. Since the denominator is $1/\sqrt{3}$, its reciprocal is $\sqrt{3}$, so

$$\frac{\sqrt{3}}{\left(\frac{1}{\sqrt{3}}\right)} = \sqrt{3} \cdot \sqrt{3} = 3$$

Using this result, we find that

$$\int_{\pi/12}^{\pi/6} \frac{1}{\tan(2\theta) \cdot \cos^2(2\theta)}\, d\theta = \left(\frac{1}{2}\right) \ln(3)$$

We can further simplify this by using the rule of logarithms which states that

$$c \ln(d) = \ln(d^c),$$

and setting c to $1/2$ and d to 3, hence we get

$$\int_{\pi/12}^{\pi/6} \frac{1}{\tan(2\theta) \cdot \cos^2(2\theta)}\, d\theta = \ln(3^{1/2}) = \ln(\sqrt{3})$$