Consider the functional

$$S[y] = \int_a^b dx \left( y^2 \sinh x - \frac{2y'^2}{\sinh x} \right), \quad y(a) = A,\ y(b) = B,\ 0 < a < b$$

(a) Find the Euler–Lagrange equation associated with this functional.

(b) Find a new independent variable, $u$, depending only on $x$, that transforms this functional to

$$S[y] = \int_{u_1}^{u_2} du \left( y^2 - 2(y'(u))^2 \right), \tag{4}$$

for some limits $u_1$ and $u_2$, which you should find.

(c) Solve the associated Euler–Lagrange equation for the functional (4).

(d) Hence or otherwise show that the solution of

$$2 \sinh x \frac{d^2 y}{dx^2} - 2 \cosh x \frac{dy}{dx} + y \sinh^3 x = 0, \quad y(1) = 0,\ y(2) = 2,$$

is

$$y = \frac{2 \sin\left( \frac{1}{\sqrt{2}}(\cosh x - \cosh 1) \right)}{\sin\left( \frac{1}{\sqrt{2}}(\cosh 2 - \cosh 1) \right)}.$$

[No marks will be given for verifying that this is a solution by direct substitution.]