10. Use the method of separation of variables to show that the wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$$

has solutions of the form

$$y = \begin{cases} \sin kx \cos kct, \\ \sin kx \sin kct, \\ \cos kx \cos kct, \\ \cos kx \sin kct. \end{cases}$$

What is a *normal mode* of a string? State two properties of normal modes that make them useful in the analysis of oscillating systems.

A string in which transverse displacements propagate at speed c has its ends fixed at $x = 0$ and $x = \pi$. The midpoint of the string is drawn aside a small distance h and gently released. Show that at any subsequent time t the displacement is

$$y = \frac{8h}{\pi^2} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^2} \sin(2r+1)x \, \cos(2r+1)ct.$$

What is the ratio of the energies contained in the fundamental and third normal modes?