Electromagnetic waves

This is an appropriate point at which to demonstrate that Maxwell's equations possess propagating wave-like solutions. Let us start from Maxwell's equations in free space (*i.e.*, with no charges and no currents):

$$\nabla\cdot\mathbf{E} = 0, \tag{430}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{431}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{432}$$

$$\nabla\times\mathbf{B} = \epsilon_0\mu_0\,\frac{\partial\mathbf{E}}{\partial t}. \tag{433}$$

Note that these equations exhibit a nice symmetry between the electric and magnetic fields.

There is an easy way to show that the above equations possess wave-like solutions, and a hard way. The easy way is to assume that the solutions are going to be wave-like beforehand. Specifically, let us search for plane-wave solutions of the form:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\cos\left(\mathbf{k}\cdot\mathbf{r}-\omega\,t\right), \tag{434}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0\cos\left(\mathbf{k}\cdot\mathbf{r}-\omega\,t+\phi\right). \tag{435}$$

Here, $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant vectors, $\mathbf{k}$ is called the wave-vector, and ω is the angular frequency. The frequency in hertz, f, is related to the angular frequency via $\omega = 2\pi\,f$. The frequency is conventionally defined to be positive. The quantity ϕ is a phase difference between the electric and magnetic fields. Actually, it is more convenient to write

$$\mathbf{E} = \mathbf{E}_0\,\mathrm{e}^{\,\mathrm{i}\,(\mathbf{k}\,\mathbf{r}-\omega\,t)}, \tag{436}$$

$$\mathbf{B} = \mathbf{B}_0\,\mathrm{e}^{\,\mathrm{i}\,(\mathbf{k}\,\mathbf{r}-\omega\,t)}, \tag{437}$$

where, by convention, the physical solution is the *real part* of the above equations. The phase difference ϕ is absorbed into the constant vector $\mathbf{B}_0$ by allowing it to become complex. Thus, $\mathbf{B}_0 \rightarrow \mathbf{B}_0\,\mathrm{e}^{\,\mathrm{i}\,\phi}$. In general, the vector $\mathbf{E}_0$ is also complex.

A wave maximum of the electric field satisfies

$$\mathbf{k}\cdot\mathbf{r} = \omega\,t + n\,2\pi + \phi, \tag{438}$$

where n is an integer and ϕ is some phase angle. The solution to this equation is a set of equally spaced parallel planes (one plane for each possible value of n), whose normals lie in the direction of the wave-vector $\mathbf{k}$, and which propagate in this direction with phase-velocity

$$v = \frac{\omega}{k}. \tag{439}$$

The spacing between adjacent planes (*i.e.*, the wave-length) is given by

$$\lambda = \frac{2\pi}{k} \tag{440}$$

(see Fig. 35).

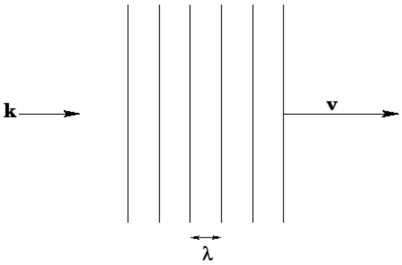

Figure 35:

Consider a general plane-wave vector field

$$\mathbf{A} = \mathbf{A}_0\,\mathrm{e}^{\,\mathrm{i}\,(\mathbf{k}\,\mathbf{r}-\omega\,t)}. \tag{441}$$

What is the divergence of $\mathbf{A}$? This is easy to evaluate. We have

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = (A_{0x}\,\mathrm{i}\,k_x + A_{0y}\,\mathrm{i}\,k_y + A_{0z}\,\mathrm{i}\,k_z)\,\mathrm{e}^{\mathrm{i}\,(\mathbf{k}\cdot\mathbf{r}-\omega t)} = \mathrm{i}\,\mathbf{k}\cdot\mathbf{A}. \quad (442)$$

How about the curl of $\mathbf{A}$? This is slightly more difficult. We have

$$(\nabla\times\mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = (\mathrm{i}\,k_y A_z - \mathrm{i}\,k_z A_y) = \mathrm{i}\,(\mathbf{k}\times\mathbf{A})_x. \quad (443)$$

This is easily generalized to

$$\nabla\times\mathbf{A} = \mathrm{i}\,\mathbf{k}\times\mathbf{A}. \quad (444)$$

We can see that vector field operations on a plane-wave simplify to replacing the ∇ operator with $\mathrm{i}\,\mathbf{k}$.

The first Maxwell equation (430) reduces to

$$\mathrm{i}\,\mathbf{k}\cdot\mathbf{E}_0 = 0, \quad (445)$$

using the assumed electric and magnetic fields (436) and (437), and Eq. (442). Thus, the electric field is perpendicular to the direction of propagation of the wave. Likewise, the second Maxwell equation gives

$$\mathrm{i}\,\mathbf{k}\cdot\mathbf{B}_0 = 0, \quad (446)$$

implying that the magnetic field is also perpendicular to the direction of propagation. Clearly, the wave-like solutions of Maxwell's equation are a type of *transverse wave*. The third Maxwell equation gives

$$\mathrm{i}\,\mathbf{k}\times\mathbf{E}_0 = \mathrm{i}\,\omega\,\mathbf{B}_0, \quad (447)$$

where use has been made of Eq. (444). Dotting this equation with $\mathbf{E}_0$ yields

$$\mathbf{E}_0\cdot\mathbf{B}_0 = \frac{\mathbf{E}_0\cdot\mathbf{k}\times\mathbf{E}_0}{\omega} = 0. \quad (448)$$

Thus, the electric and magnetic fields are mutually perpendicular. Dotting equation (447) with $\mathbf{B}_0$ yields

$$\mathbf{B}_0\cdot\mathbf{k}\times\mathbf{E}_0 = \omega\,B_0^{\,2} > 0. \quad (449)$$

Thus, the vectors $\mathbf{E}_0$, $\mathbf{B}_0$, and $\mathbf{k}$ are mutually perpendicular, and form a right-handed set. The final Maxwell equation gives

$$\mathrm{i}\,\mathbf{k}\times\mathbf{B}_0 = -\mathrm{i}\,\epsilon_0\mu_0\,\omega\,\mathbf{E}_0. \quad (450)$$

Can you show how you derive (450) step by step from the maxwell equation that they use. Please use vector notation in the derivation.