http://ramanujan.math.trinity.edu/wtrench/texts/TRENCH_LAGRANGE_MULTIPLIERS.PDF

the extrema subject to two constraints is on page 11 in the link above

5 Extrema subject to two constraints

Here is Theorem 1 with $m = 2$.

Theorem 3 *Suppose that $n > 2$. If $\mathbf{X}_0$ is a local extreme point of f subject to $g_1(\mathbf{X}) = g_2(\mathbf{X}) = 0$ and*

$$\begin{vmatrix} \frac{\partial g_1(\mathbf{X}_0)}{\partial x_r} & \frac{\partial g_1(\mathbf{X}_0)}{\partial x_s} \\ \frac{\partial g_2(\mathbf{X}_0)}{\partial x_r} & \frac{\partial g_2(\mathbf{X}_0)}{\partial x_s} \end{vmatrix} \neq 0 \tag{19}$$

for some r and s in $\{1, 2, \ldots, n\}$, then there are constants λ and μ such that

$$\frac{\partial f(\mathbf{X}_0)}{\partial x_i} - \lambda \frac{\partial g_1(\mathbf{X}_0)}{\partial x_i} - \mu \frac{\partial g_2(\mathbf{X}_0)}{\partial x_i} = 0, \tag{20}$$

$1 \le i \le n$.

Proof For notational convenience, let $r = 1$ and $s = 2$. Denote

$$\mathbf{U} = (x_3, x_4, \ldots x_n) \quad \text{and} \quad \mathbf{U}_0 = (x_{30}, x_{30}, \ldots x_{n0}).$$

Since

$$\begin{vmatrix} \frac{\partial g_1(\mathbf{X}_0)}{\partial x_1} & \frac{\partial g_1(\mathbf{X}_0)}{\partial x_2} \\ \frac{\partial g_2(\mathbf{X}_0)}{\partial x_1} & \frac{\partial g_2(\mathbf{X}_0)}{\partial x_2} \end{vmatrix} \neq 0, \tag{21}$$

the Implicit Function Theorem (Theorem 6.4.1, p. 420) implies that there are unique continuously differentiable functions

$$h_1 = h_1(x_3, x_4, \ldots, x_n) \quad \text{and} \quad h_2 = h_1(x_3, x_4, \ldots, x_n),$$

defined on a neighborhood $N \subset \mathbb{R}^{n-2}$ of $\mathbf{U}_0$, such that $(h_1(\mathbf{U}), h_2(\mathbf{U}), \mathbf{U}) \in D$ for all $\mathbf{U} \in N$, $h_1(\mathbf{U}_0) = x_{10}$, $h_2(\mathbf{U}_0) = x_{20}$, and

I don't get how they can take the derivative of g_1 and g_2 with respect to x_1 *and* x_2 when they are defined as

$$h_1 = h_1(x_3, x_4, \ldots, x_n) = x_1 \ \ and \ \ h_2 = h_2(x_3, x_4, \ldots, x_n) = x_2$$

I need a mathematical justification for how this can be written simply as (21) when x_1 *and* x_2 are defined as functions from other variables: $x_3, x_4, \ldots, x_n$

Is it by using the chain rule or something else? Please og through the detailed steps for how this can be a normal quadratic matrix as defined in the implicit function theorem and by that showing why this is an appliable form of the implicit function theorem nonsingular matrix in (21):

Note:

The function G has image in $\mathbb{R}^2$, so we call the two components h_1 and h_2, that is, $G(Y) = (h_1(Y), h_2(Y))$ where $h_1, h_2 : \mathbb{R}^{n-2} \to \mathbb{R}$. For us $Y = (x_3, \ldots, x_n)$, therefore both functions h_1 and h_2 depend on $(x_3, \ldots, x_n)$ and we write that by saying $h_1 = h_1(x_3, \ldots, x_n)$ and $h_2 = h_2(x_3, \ldots, x_n)$ (there is a typo there in the book, certainly it is not necessarily true that $h_2 = h_1$). The two functions h_1, h_2 are defined on the neighborhood N of Y_0 in $\mathbb{R}^{n-2}$.

Now, from (3), for all $\mathbf{Y} \in N$, $(G(\mathbf{Y}), \mathbf{Y}) = (h_1(\mathbf{Y}), h_2(\mathbf{Y}), \mathbf{Y}) \in M$. As $M \subseteq D$ we have $(h_1(\mathbf{Y}), h_2(\mathbf{Y}), \mathbf{Y}) \in D$.

Theorem 6.4.1 (The Implicit Function Theorem) *Suppose that* $\mathbf{F} : \mathbb{R}^{n+m} \to \mathbb{R}^m$ *is continuously differentiable on an open set* S *of* $\mathbb{R}^{n+m}$ *containing* $(\mathbf{X}_0, \mathbf{U}_0)$. *Let* $\mathbf{F}(\mathbf{X}_0, \mathbf{U}_0) = \mathbf{0}$, *and suppose that* $\mathbf{F}_\mathbf{U}(\mathbf{X}_0, \mathbf{U}_0)$ *is nonsingular. Then there is a neighborhood* M *of* $(\mathbf{X}_0, \mathbf{U}_0)$, *contained in* S, *on which* $\mathbf{F}_\mathbf{U}(\mathbf{X}, \mathbf{U})$ *is nonsingular and a neighborhood* N *of* $\mathbf{X}_0$ *in* $\mathbb{R}^n$ *on which a unique continuously differentiable transformation* $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^m$ *is defined, such that* $\mathbf{G}(\mathbf{X}_0) = \mathbf{U}_0$ *and*

Search for the theorem in this link if you want to see more from the implicit function theorem. It is on page 420

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