1. Mean-field theory. Consider the Ising model on a 2-D square lattice with N sites. Each sites has a spin $s_i = \pm 1$. The Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j, \qquad (1)$$

where $\langle i,j \rangle$ means summing over nearest neighbors, and J is the coupling strength, which has the dimension of energy. Now, assuming that four neighbors of each spin are magnetized with the average magnetization $m = \langle s_i \rangle$, show that the *mean-field* Hamiltonian becomes

$$\mathcal{H}_{mf} = -4JNm^2 - 4Jm \sum_j s_j. \qquad (2).$$

a. At temperature T, what is the value for $\langle s_j \rangle$, given m? At what temperature T_c is the phase transition, in mean field theory? Show that $m \propto (T_c - T)^\beta$. What is β?

b. Obtain the Helmholtz free energy in terms of m, temperature, and J. On a a single graph, plot this free energy for $1/(k_B T) = 0.1, 0.25$, and 0.5, for $-2 < m < 2.$, showing the continuous phase transition, assuming that $J = 1$. Show that the solution in **a** corresponds to the minimum of the free energy in terms of m.

c. What would the mean-field Hamiltonian be for the square-lattice Ising model in an external field H? What is the corresponding Helmholtz free energy, in terms of m and H? On a single graph, plot this free energy for $1/(k_B T) = 0.5$ and $H = 0$, 0.5, 1.0, and 1.5, showing metastability and an abrupt transition. At what value of H does the metastable state become completely unstable?