Helmholtz's theorem

Let us now embark on a slight mathematical digression. Up to now, we have only studied the electric and magnetic fields generated by stationary charges and steady currents. We have found that these fields are describable in terms of four field equations:

$$\nabla \cdot \mathbf{E} \quad = \quad \frac{\rho}{\epsilon_0}, \tag{279}$$

$$\nabla \times \mathbf{E} \quad = \quad 0 \tag{280}$$

for electric fields, and

$$\nabla \cdot \mathbf{B} \quad = \quad 0, \tag{281}$$

$$\nabla \times \mathbf{B} \quad = \quad \mu_0 \, \mathbf{j} \tag{282}$$

for magnetic fields. There are no other field equations. This strongly suggests that if we know the divergence and the curl of a vector field then we know everything there is to know about the field. In fact, this is the case. There is a mathematical theorem which sums this up. It is called *Helmholtz's theorem* after the German polymath Hermann Ludwig Ferdinand von Helmholtz.

Let us start with scalar fields. Field equations are a type of differential equation: *i.e.*, they deal with the infinitesimal differences in quantities between neighbouring points. The question is, what differential equation completely specifies a scalar field? This is easy. Suppose that we have a scalar field ϕ and a field equation which tells us the gradient of this field at all points: something like

$$\nabla \phi = \mathbf{A}, \tag{283}$$

where $\mathbf{A}(\mathbf{r})$ is a vector field. Note that we need $\nabla \times \mathbf{A} = 0$ for self consistency, since the curl of a gradient is automatically zero. The above equation completely specifies ϕ once we are given the value of the field at a single point, P (say). Thus,

$$\phi(Q) = \phi(P) + \int_P^Q \nabla \phi \cdot d\mathbf{l} = \phi(P) + \int_P^Q \mathbf{A} \cdot d\mathbf{l}, \tag{284}$$

where Q is a general point. The fact that $\nabla \times \mathbf{A} = 0$ means that $\mathbf{A}$ is a conservative field, which guarantees that the above equation gives a unique value for ϕ at a general point in space.

Suppose that we have a vector field $\mathbf{F}$. How many differential equations do we need to completely specify this field? Hopefully, we only need two: one giving the divergence of the field, and one giving its curl. Let us test this hypothesis. Suppose that we have two field equations:

$$\nabla \cdot \mathbf{F} = D, \tag{285}$$

$$\nabla \times \mathbf{F} = \mathbf{C}, \tag{286}$$

where D is a scalar field and $\mathbf{C}$ is a vector field. For self-consistency, we need

$$\nabla \cdot \mathbf{C} = 0, \tag{287}$$

since the divergence of a curl is automatically zero. The question is, do these two field equations plus some suitable boundary conditions completely specify $\mathbf{F}$? Suppose that we write

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}. \tag{288}$$

In other words, we are saying that a general field $\mathbf{F}$ is the sum of a conservative field, ∇U, and a solenoidal field, $\nabla \times \mathbf{W}$. This sounds plausible, but it remains to be proved. Let us start by taking the divergence of the above equation, and making use of Eq. (285). We get

$$\nabla^2 U = -D. \tag{289}$$

Note that the vector field $\mathbf{W}$ does not figure in this equation, because the divergence of a curl is automatically zero. Let us now take the curl of Eq. (288):

$$\nabla \times \mathbf{F} = \nabla \times \nabla \times \mathbf{W} = \nabla(\nabla \cdot \mathbf{W}) - \nabla^2 \mathbf{W} = -\nabla^2 \mathbf{W}. \tag{290}$$

Here, we assume that the divergence of $\mathbf{W}$ is zero. This is another thing which remains to be proved. Note that the scalar field U does not figure in this equation, because the curl of a divergence is automatically zero. Using Eq. (286), we get

$$\nabla^2 W_x = -C_x, \tag{291}$$

$$\nabla^2 W_y = -C_y, \tag{292}$$

$$\nabla^2 W_z = -C_z, \tag{293}$$

So, we have transformed our problem into four differential equations, Eq. (289) and Eqs. (291)-(293), which we need to solve. Let us look at these equations. We immediately notice that they all have exactly the same form. In fact, they are all versions of Poisson's equation. We can now make use of a principle made famous by Richard P. Feynman: ``the same equations have the same solutions.'' Recall that earlier on we came across the following equation:

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0},$$ (294)

where ϕ is the electrostatic potential and ρ is the charge density. We proved that the solution to this equation, with the boundary condition that ϕ goes to zero at infinity, is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'.$$ (295)

Well, if the same equations have the same solutions, and Eq. (295) is the solution to Eq. (294), then we can immediately write down the solutions to Eq. (289) and Eqs. (291)-(293). We get

$$U(\mathbf{r}) = \frac{1}{4\pi} \int \frac{D(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}',$$ (296)

and

$$W_x(\mathbf{r}) = \frac{1}{4\pi} \int \frac{C_x(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}',$$ (297)

$$W_y(\mathbf{r}) = \frac{1}{4\pi} \int \frac{C_y(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}',$$ (298)

$$W_z(\mathbf{r}) = \frac{1}{4\pi} \int \frac{C_z(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'.$$ (299)

The last three equations can be combined to form a single vector equation:

$$\mathbf{W}(\mathbf{r}) = \frac{1}{4\pi} \int \frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'.$$ (300)

We assumed earlier that $\nabla \cdot \mathbf{W} = 0$. Let us check to see if this is true. Note that

$$\frac{\partial}{\partial x}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = -\frac{x - x'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{x' - x}{|\mathbf{r} - \mathbf{r}'|^3} = -\frac{\partial}{\partial x'}\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right),$$ (301)

which implies that

$$\nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) = -\nabla'\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right),$$ (302)

where ∇' is the operator $(\partial/\partial x', \partial/\partial y', \partial/\partial z')$. Taking the divergence of Eq. (300), and making use of the above relation, we obtain

$$\nabla \cdot \mathbf{W} = \frac{1}{4\pi} \int \mathbf{C}(\mathbf{r}') \cdot \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) d^3\mathbf{r}' = -\frac{1}{4\pi} \int \mathbf{C}(\mathbf{r}') \cdot \nabla'\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) d^3\mathbf{r}'.$$ (303)

Now

$$\int_{-\infty}^{\infty} g\frac{\partial f}{\partial x}\, dx = [gf]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f\frac{\partial g}{\partial x}\, dx.$$ (304)

However, if $g f \to 0$ as $x \to \pm\infty$ then we can neglect the first term on the right-hand side of the above equation and write

$$\int_{-\infty}^{\infty} g\frac{\partial f}{\partial x}\, dx = -\int_{-\infty}^{\infty} f\frac{\partial g}{\partial x}\, dx.$$ (305)

A simple generalization of this result yields

$$\int \mathbf{g} \cdot \nabla f\, d^3\mathbf{r} = -\int f\, \nabla \cdot \mathbf{g}\, d^3\mathbf{r},$$ (306)

provided that $g_x f \to 0$ as $|\mathbf{r}| \to \infty$, *etc.* Thus, we can deduce that

$$\nabla \cdot \mathbf{W} = \frac{1}{4\pi} \int \frac{\nabla' \cdot \mathbf{C}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$ (307)

from Eq. (303), provided $|\mathbf{C}(\mathbf{r})|$ is bounded as $|\mathbf{r}| \to \infty$. However, we have already shown that $\nabla \cdot \mathbf{C} = 0$ from self-consistency arguments, so the above equation implies that $\nabla \cdot \mathbf{W} = 0$, which is the desired result.

We have constructed a vector field $\mathbf{F}$ which satisfies Eqs. (285) and (286) and behaves sensibly at infinity: *i.e.*, $|\mathbf{F}| \to 0$ as $|\mathbf{r}| \to \infty$. But, is our solution the only possible solution of Eqs. (285) and (286) with sensible boundary conditions at infinity? Another way of posing this question is to ask whether there are any solutions of

$$\nabla^2 U = 0, \quad \nabla^2 W_i = 0,$$ (308)

How do they come up with the equations in (308) mathematically? Why do (308) give solutions to (285) and (286). Or why do (308) determine whether (285) and (286) have 1 or more solutions? I

where i denotes x, y, or z, which are bounded at infinity. If there are then we are in trouble, because we can take our solution and add to it an arbitrary amount of a vector field with zero divergence and zero curl, and thereby obtain another solution which also satisfies physical boundary conditions. This would imply that our solution is not unique. In other words, it is not possible to unambiguously reconstruct a vector field given its divergence, its curl, and physical boundary conditions. Fortunately, the equation

$$\nabla^2 \phi = 0, \tag{309}$$

which is called *Laplace's equation*, has a very nice property: its solutions are *unique*. That is, if we can find a solution to Laplace's equation which satisfies the boundary conditions then we are guaranteed that this is the *only* solution. We shall prove this later on in the course. Well, let us invent some solutions to Eqs. (308) which are bounded at infinity. How about

$$U = W_i = 0? \tag{310}$$