Theorem

Let $\mathbf{T}_n$ be a triangular matrix (either upper or lower) of order n.

Let $\det(\mathbf{T}_n)$ be the determinant of $\mathbf{T}_n$.

Then $\det(\mathbf{T}_n)$ is equal to the product of all the diagonal elements of $\mathbf{T}_n$.

That is:

$$\det(\mathbf{T}_n) = \prod_{k=1}^{n} a_{kk}$$

I wanted a proof for this not by induction but by doing the actual expansion. And the only property ofmatrixes that you can use it that interchanging any rows only changes the sign of the determinant. You have to expand the matrix along its first row. Here is my attempt. A problem that I got is written in red below

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,k-1} & a_{1,k} \\ 0 & a_{22} & a_{23} & \cdots & a_{2,k-1} & a_{2,k} \\ 0 & 0 & a_{33} & \cdots & a_{3,k-1} & a_{3,k} \\ 0 & 0 & 0 & \cdots & a_{4,k-1} & a_{4,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{kk} \end{vmatrix}$$

$$a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} & \cdots & a_{2,k-1} & a_{2,k} \\ 0 & a_{33} & a_{34} & \cdots & a_{3,k-1} & a_{3,k} \\ 0 & 0 & a_{44} & \cdots & a_{4,k-1} & a_{4,k} \\ 0 & 0 & 0 & \cdots & a_{5,k-1} & a_{5,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{kk} \end{vmatrix}$$

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$$-a_{12}\begin{vmatrix} 0 & a_{23} & a_{24} & \cdots & a_{2,k-1} & a_{2,k} \\ 0 & a_{33} & a_{34} & \cdots & a_{3,k-1} & a_{3,k} \\ 0 & 0 & a_{44} & \cdots & a_{4,k-1} & a_{4,k} \\ 0 & 0 & 0 & \cdots & a_{5,k-1} & a_{5,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{kk} \end{vmatrix}$$

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$$+a_{1k}\begin{vmatrix} 0 & a_{22} & a_{23} & \cdots & a_{2,k-2} & a_{2,k-1} \\ 0 & 0 & a_{33} & \cdots & a_{3,k-2} & a_{3,k-1} \\ 0 & 0 & 0 & \cdots & a_{4,k-2} & a_{4,k-1} \\ 0 & 0 & 0 & \cdots & a_{5,k-2} & a_{5,k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix}$$

For the last determinant since one row is a 0 row we know from above that since we can expand along any row it is 0

$$a_{1k}\begin{vmatrix} 0 & a_{22} & a_{23} & \cdots & a_{2,k-2} & a_{2,k-1} \\ 0 & 0 & a_{33} & \cdots & a_{3,k-2} & a_{3,k-1} \\ 0 & 0 & 0 & \cdots & a_{4,k-2} & a_{4,k-1} \\ 0 & 0 & 0 & \cdots & a_{5,k-2} & a_{5,k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = 0$$

For the others except the one with cofactor a_{11}

Since we know that we can swap any rows to calculate determinant and only changing sign. For each det that has a 0 in its first column must be 0. For example for this one

$$-a_{12}\begin{vmatrix} 0 & a_{23} & a_{24} & \cdots & a_{2,k-1} & a_{2,k} \\ 0 & a_{33} & a_{34} & \cdots & a_{3,k-1} & a_{3,k} \\ 0 & 0 & a_{44} & \cdots & a_{4,k-1} & a_{4,k} \\ 0 & 0 & 0 & \cdots & a_{5,k-1} & a_{5,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{kk} \end{vmatrix}$$

We swap row k and 1

$$a_{12}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & a_{k,k} \\ 0 & a_{33} & a_{34} & \cdots & a_{3,k-1} & a_{3,k} \\ 0 & 0 & a_{44} & \cdots & a_{4,k-1} & a_{4,k} \\ 0 & 0 & 0 & \cdots & a_{5,k-1} & a_{5,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{23} & a_{24} & \cdots & a_{2,k-1} & a_{2k} \end{vmatrix}$$

$$= a_{12}a_{k,k}\begin{vmatrix} 0 & a_{33} & a_{34} & \cdots & a_{3,k-2} & a_{3,k-1} \\ 0 & 0 & a_{44} & \cdots & a_{4,k-2} & a_{4k-1} \\ 0 & 0 & 0 & \cdots & a_{5,k-2} & a_{5,k-1} \\ 0 & 0 & 0 & \cdots & a_{6,k-2} & a_{6,k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{23} & a_{24} & \cdots & a_{2,k-2} & a_{2,k-1} \end{vmatrix}$$

We know that row k-1 is 0 in all elements except the last nad repeat the process with$a_{k-1,k-1}$ as the new cofactor

$$= a_{12}a_{k,k}\begin{vmatrix} 0 & a_{33} & a_{34} & \cdots & a_{3,k-2} & a_{3,k-1} \\ 0 & 0 & a_{44} & \cdots & a_{4,k-2} & a_{4k-1} \\ 0 & 0 & 0 & \cdots & a_{5,k-2} & a_{5,k-1} \\ 0 & 0 & 0 & \cdots & a_{6,k-2} & a_{6,k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{k-1,k-1} \\ 0 & a_{23} & a_{24} & \cdots & a_{2,k-2} & a_{2,k-1} \end{vmatrix}$$

$$= -a_{12}a_{k,k}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & a_{k-1,k-1} \\ 0 & 0 & a_{44} & \cdots & a_{4,k-2} & a_{4k-1} \\ 0 & 0 & 0 & \cdots & a_{5,k-2} & a_{5,k-1} \\ 0 & 0 & 0 & \cdots & a_{6,k-2} & a_{6,k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{33} & a_{34} & \cdots & a_{3,k-2} & a_{3,k-1} \\ 0 & a_{23} & a_{24} & \cdots & a_{2,k-2} & a_{2,k-1} \end{vmatrix}$$

$$= a_{12}a_{k,k}a_{k-1,k-1}\begin{vmatrix} 0 & 0 & a_{44} & \cdots & a_{4,k-3} & a_{4k-2} \\ 0 & 0 & 0 & \cdots & a_{5,k-3} & a_{5,k-2} \\ 0 & 0 & 0 & \cdots & a_{6,k-3} & a_{6,k-2} \\ 0 & 0 & 0 & \cdots & a_{7,k-3} & a_{7,k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{33} & a_{34} & \cdots & a_{3,k-3} & a_{3,k-2} \\ 0 & a_{23} & a_{24} & \cdots & a_{2,k-3} & a_{2,k-2} \end{vmatrix}$$

I could have repeated this but I don’t know what happens with the last rows since I keep taking a row that is higher and higher up in the matrix rows and therefore has more and more rows in the bottom as illustrated by the original rows 3 and 2 that are in the bottom above in red