PROBABILITY

Probability is the branch of mathematics that investigates the long term behavior of random events. That may seem absurd - how can one intelligently discuss random events? The point is that, in mathematics, a random phenomenon is one whose individual outcomes are unpredictable, but whose long term behavior, i.e. the outcomes of many individual events, reveals definite patterns. As an example, the outcome (heads or tails) of a single toss of a fair coin cannot be predicted. However, because the coin is fair, the proportion of heads to tails in 1000 tosses will be very close to 1:1.

Many aspects of daily life are random in the above sense: life span, occurrence of cancer, driving of an automobile involved in an accident are some examples of interest to insurance companies. The outcomes of casino gambling games and state lotteries are random, but considerable revenue is derived from their long term predictability. We'll investigate some methods for calculating probabilities, apply them to study certain gambling games, and learn about an application of probability to the process by which new therapies for dread diseases are studied.

Exercise 1. *In your group of 4 students, pair up and designate one person as tosser and the other as recorder. The tosser will toss a coin 50 times and the recorder will record the outcome (H or T). Now switch roles for another 50 tosses. Then reform your pairs and repeat the tossing so that you end up with 200 outcomes. What fraction of tosses resulted in H? What fraction resulted in T? Calculate the ratio of the number of H's to the number of T's . Incidentally, how many runs of 3 or more H's occurred? 4 or more?*

Exercise 2. *Suppose that we toss three fair coins. Write the outcome that all fall on heads as (H,H,H). If the first falls on T and the second two on H, write this as (T,H,H) and note that this is different from (H,H,T). With this notation, list all possible outcomes. Now list the all of the outcomes in which all 3 coins fall on the same side, and those in which two fall on the same side.*

It's useful to have some terminology. Let's use the term "experiment" for a process we are investigating. For example, each coin toss can be thought of as an experiment. In the first exercise each coin toss is an experiment with two possible outcomes H or T. In the second exercise each tossing of three coins is an experiment and the resulting triple of H's and T's is an outcome.

(1) The **sample space** for an experiment is the set of all possible outcomes: The sample space for a single coin toss is {H,T}. For tossing three coins it is the eight element set of the previous exercise. For win, place, and show finishers in a race with 11 horses, it would be the set of all ordered lists of 3 out of the 11 horses. For example, the following 11 horses ran the 2012 Belmont Stakes: Dullahan, I'll Have Another, Paynter, Alpha, Union Rags, Street Life, Optimizer, Atigun, Five Sixteen,Guyana Star Dweej, and My Adonis. The sample space consists of three tuples of horses like (Dullahan, I'll Have Another, Paynter), (Alpha, Street Life, Five Sixteen). And (Dullahan, I'll Have Another, Paynter) is different from (Paynter, Dullahan, I'll Have Another) so there are lots of elements in the sample space.

(2) An **event** is any collection of outcomes, i.e. a subset of the sample space. In our example of tossing 3 fair coins, the event that two coins fall on heads is {(H,H,T), (H,T,H), (T,H,H)}. The event that Union Rags, Street Life, and Optimizer finish in the money is {(Union Rags, Street Life, Optimizer), (Union Rags, Optimizer,Street Life), (Optimizer,Street Life, Union Rags), (Optimizer, Union Rags, Street Life), (Street Life, Union Rags, Optimizer), (Street Life, Optimizer, Union Rags)}.

Definition 1. *The **probability** of an event is the proportion of times the event would occur if we repeat the experiment many times.*

An equivalent concept that is often used for gambling purposes but that we will not use is

Definition 2. *The **odds** of any event is the ratio of the probability of that event to the probability that event doesn't occur. In case of tossing a single fair coin, the odds of H is the ratio of the number of H's to the number of T's (because these are the only 2 possible outcomes).*

These definitions, which can be made mathematically sound, are vague in a practical sense. How long is long enough in a "very long sequence of repetitions"? Nevertheless, we can derive some information about probabilities from the definition.

(1) The probability of any event is a real number between 0 and 1. An event with probability 0 never occurs and an event with probability 1 must occur.

(2) We're only going to discuss phenomena with only finitely many outcomes. Since in any repetition there is some outcome, the sum of the probabilities of all possible outcomes is equal to 1.

(3) If two events have no outcomes in common, then the probability that one or the other occurs is the sum of their individual probabilities. For example, suppose that a jar contains red balls, blue balls, and green balls, and that the probability of drawing a red ball is .2, a blue ball .45, and green ball .35. This means that the more times we run the experiment of picking a ball at random, noting its color, and replacing the ball, the proportion of times we draw a red ball gets closer to 20%, the propoertion of times we get a blue ball is closer to 45% and for a green ball the closer it gets to 35%. Then the probability of picking either a red ball or a blue ball is .65, since in a very long sequence of repetitionsthat happens close to 20%+45% =65% of the time.

Exercise 3. *What is wrong with the following weather prediction (actually heard broadcast from Richmond , VA in the 1980's)? The probability of rain on Saturday is 50% and the same is true for Sunday. So carry an umbrella this weekend because is rain is certain. List the outcomes sample space as (R,N) meaning rain on Saturday and no rain on Sunday. What outcomes are in the event of rain sometime this weekend? What outcome(s) are missing? Why is condition 3 just above not violated?*

A probability model is a description of a random phenomenon that consists of a sample space and an assignment of probabilities to events. A simple but important probability model is one which assigns equal probability to all outcomes, e.g. the model we used in our coin tossing examples. Similarly, on a fair die (one of a pair of dice), each of the six numbers on the faces is equally likely to show up.

Exercise 4. (1) *Assuming that all outcomes for rolling two dice have equal probability, list the sample space.*
(2) *Calculate the probability that numbers showing up are equal.*
(3) *Calculate the probability that the sum of the numbers showing up is even. Do this by adding the probabilities of the various ways that the sum of the numbers showing up is even.*

Generalizing what you observed in this example, we have:
If a random phenomenon has k possible equally likely outcomes, then
(1) *The probability of any outcome is* $\frac{1}{k}$.
(2) *If A is an event then the probability of* $A = \frac{\text{number of outcomes in A}}{k}$.

In order to get some interesting examples in the context of equally likely outcomes, we need to learn how to calculate the numerator and denominator. Here are two laws for counting:

(1) **Law of Sums:** Suppose a procedure can be completed by exactly one of two tasks. If there are n ways to complete task 1 and m ways to complete task 2, then there are n+m ways to complete the procedure. As an example, suppose that the chair of a combined administration, faculty, and student committee needs to be selected, but the chair needs to be either a student or a faculty member. If there are 18 faculty members and 8 students on the committee then there are $18 + 8 = 26$ ways to choose the chair.

(2) **Law of Products:** Suppose a procedure can only be completed by both of two tasks. If there are n ways to complete task 1 and m ways to complete task 2, then there are $n \times m$ ways to complete the procedure. For example suppose that a chair and secretary for our committee need to be selected. If the chair must be a faculty member and the secretary a student, then there are $18 \times 8 = 144$ possible (chair,secretary) combinations.

Exercise 5. *Reformulate the Law of Sums for a procedure that can be accomplished by exactly one of n tasks (k_1 ways to accomplish task 1, k_2 ways to accomplish task 2, etc.), and the Law of Products for a procedure that requires each of n tasks.*

Permutations

Now let's elaborate. A permutation of k objects is simply an ordering of them. For example, there are 2 ways to order the symbols a, b :

$$(a, b) \text{ and } (b, a).$$

If you think of a permutation of two objects as a procedure requiring two tasks (who comes first and who comes second) then the law of products tells you that there are exactly two permutations (two ways to decide who comes first and 1 way to decide who comes second).

Exercise 6. *How many permutations are there of 3 objects? 4 objects?*

Notation 1. *The symbol $n!$ (read n factorial) denotes the number of permutations of n objects.*

In a horse race, the first three finishers are referred to as win, place, and show. So if 8 horses run a race, a win, place, show outcome is a permutation of 3 of the 8 horses. We refer to this as a permutation of 8 horses taken 3 at a time. Assuming no ties, there are 8 possibilities for win, 7 for place, and 6 for show, so by the law of products, 336 such outcomes.

Exercise 7. *A club of 40 members needs an executive board consisting of a president, vice president, secretary, and treasurer, and no one can hold more than one of these positions. How many different executive boards are possible?*

Exercise 8. *Interpret the New Mexico Pick 3 lottery game in terms of permutations. Calculate the probability of selecting the correct number. What is the probability that the winning number has consecutive digits (consider (9,0,1) as having consecutive digits)?*

Exercise 9. *Use factorial notation for express the number of permutations of n objects taken k at a time as a fraction.*

Combinations

If we compare the Powerball game with the Pick 3, we see a major difference. Except for the Powerball number itself, we don't care about the order of the other 5 numbers, it's only the set of numbers that determine a winner. An unordered selection of k objects from n objects is called a combination of n objects taken k at a time. For example, to select one object from $\{a, b\}$ we can select either a or b. So there are 2 combinations of 2 objects taken 1 at a time. To select 2 objects from $\{a, b, c\}$ we have 3 possibilities:

$$\{a, b\}, \{a, c\}, \{b, c\}.$$

Note that selecting two objects from 3 is the same as not selecting 1 object, or to put it another way, to select 1 object not to select.

Notation 2. *The symbol* $\binom{n}{k}$ *is the number of combinations of n objects taken k at a time.*

Exercise 10. *By enumerating all the possible combinations of $\{a, b, c, d\}$, determine* $\binom{4}{1}, \binom{4}{2}, \binom{4}{3}$.

Since $\binom{4}{2}$ *is the number of ways to select 2 objects from $\{a, b, c, d\}$ let's list them out systematically starting with those selections including a : $\{a, b, \{a, c\}, \{a, d\}$. Next those including b that weren't already listed: $\{b, c\}, \{b, d\}$. I didn't write $\{b, a\}$ because here we are only interested in the objects, not how they are ordered. Finally, there is only $\{c, d\}$ for a total of* $6 = \binom{4}{2}$.

$\binom{n}{k}$ occurs in many contexts, so there must be a better way than enumeration to calculate it! Look at forming a **permutation** of n objects taken k at a time as a procedure requiring both of two tasks: task one is to select the k objects from n (i.e. forming a combination) and the second is to permute these k objects. The first task can be accomplished in $\binom{n}{k}$ ways and the second in $k!$ ways. Using Exercise 9 above, we have the equation

$$\frac{n!}{(n-k)!} = \binom{n}{k} k!$$

so that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Observe from this last equation that $\binom{n}{k} = \binom{n}{n-k}$. In other words, selecting k objects is the same as selecting $n-k$ objects not to select.

Exercise 11. *Calculate* $\binom{6}{2}, \binom{6}{3}, \binom{10}{7}$.

Exercise 12. *Estimate the probability of selecting the correct powerball number.*

Exercise 13. *Estimate the probability of being dealt a full house from a fair and fully shuffled deck of 52 cards.*

Here is a silly but useful example of permutations and combinations. Lets view any sequence of English letters as a word, whether or not it's pronounceable or has any meaning, e.g. SCVOR is a word. Clearly if we have n different letters (so $n \leq 26$) there are $n!$ different words that can be written with these letters. But what about a word like MISSISSIPPI which has multiple occurrences of the same letter? Imagine that the repeated letters are actually different: $\mathrm{MI_1S_1S_2I_2S_3S_4I_3P_1P_2I_4}$. Since we now have 11 different letters, there are 11! words using them. Interpret the construction of such a word as a procedure with several tasks. Task 1 is to construct a word where the repeated letters are not distinct (which is what we want to count), task 2 is to distribute 1,2,3,4 among the I's (there are 4! ways to do this), task 3 is to distribute 1,2,3,4 among the S's (again 4! ways to do this), and finally distribute 1,2 among the P's ($2! = 2$ ways to do this). Thus by the law of products,

$$11! = (\#\text{ ways to accomplish task 1}) \times 4!4!2!$$

so that

$$(\text{\# ways to accomplish task 1}) = \frac{11!}{4!4!2!}.$$

Exercise 14. *How many words can be spelled with the letters in philippic?*

Here is an example of a *combinatorial argument* in which something that is apparently difficult to count is shown to be the same as something easy to count. Consider an ice cream shop that sells 12 different flavors of ice cream. How many different orders of 3 scoops can be made? Here we allow all three scoops to be of the same flavor or different. Use |'s to denote the flavors of ice cream and *'s to separate the orders, so the "word"

$$|| \ast \ast ||||||| \ast ||$$

represents an order of 2 scoops of flavor 3 and 1 scoop of flavor 10. Since we put the *'s to the left of the flavor, we don't need a bar for flavor 12. For example an order of one scoop of flavor 1 and 2 scoops of flavor 12 would be represented by

$$\ast ||||||||||| \ast \ast.$$

Convince yourself that each such word written with 3 stars and 11 bars represents a different order and that every order can be represented this way.

Exercise 15. *How many possible 3 scoop order are there?*

Exercise 16. *How many 2 scoop orders are there? How many orders of at most 3 scoops are there (use the law of sums).*

Exercise 17. *Assuming that all flavors are equally likely what is the probability that someone who orders 3 scoops will order two of the same flavor? All three of the same flavor? All three of different flavor?*

Independence

A very important concept in probability is that of independence. Two events, call them A and B, are independent if the probability that they both occur is the product of their probabilities. Symbolically, write $P(A)$ and $P(B)$ for the probabilities of A and B, and $A \cap B$ for the event that both A and B occur. So A and B, are independent if

$$P(A \cap B) = P(A)P(B) \ .$$

For example, when we toss two fair coins the events $A =$"the first coin lands on heads" and $B =$"the second coin lands on tails" are independent: Because of the 4 possible outcomes, there are 2 in A: (H,H) and (H,T), so $P(A) = \frac{1}{2}$. Similarly, $P(A) = \frac{1}{2}$, but $A \cap B$ just consists of the single outcome (H,T), so that

$$P(A \cap B) = \frac{1}{4} = P(A)P(B).$$

On the other hand, if A="exactly one head" and B="exactly one tail" then these events are not independent. Of the 4 possible outcomes, exactly 2 have exactly one head (H,T) and (T,H), so $P(A) = \frac{1}{2}$. For the same reason, $P(B) = \frac{1}{2}$. But $A \cap B$ is precisely the same set of outcomes, so

$$\begin{aligned} P(A \cap B) &= \frac{1}{2} \\ &\neq P(A)P(B). \end{aligned}$$

Exercise 18. *List the possible outcomes of rolling a die. Which of the following pairs of events are independent:*

- Rolling an even number and rolling a multiple of 3?
- Rolling an even number and rolling a multiple of 4?

Exercise 19. *Many people experience respiratory difficulties in Las Cruces the amount of particulate matter in the air brought on by the frequent high winds exceeds a certain level. Similar problems occur when pollen counts can exceed certain levels . Suppose that the local air quality monitoring system finds that on* 30% *of days particulate matter is above the threshold, on* 20% *of days pollen is above the threshold and on* 6% *of days both are above their thresholds. Are the events "Particulate matter above threshold" and "pollen count above threshold" independent?*

An Application to Therapeutics

The process by which a new therapy for the treatment of a deadly illness is approved is extremely involved. It is necessarily so because the therapies, which usually destroy invading cells or interfere with natural bodily processes, are toxic. A therapy can be a drug, radiation regime, chemotherapy, or some combination thereof. Before a proposed therapy is subjected to controlled experiments on human subjects, researchers determine the Maximum Tolerated Dose or MLD. This is the dosage beyond which toxic effects outweigh any possible therapeutic benefit. Once the MLD has been established, the researcher needs to know whether the proposed therapy has sufficient likelihood for success as a treatment to warrant further testing. Naturally, since treatments are toxic and the experimental subjects are human, researchers are obligated to use the fewest number of patients as possible.

In the 1960's the biostatistician Ed Gehan proposed a simple test, based only on the concept of independence, to determine whether a proposed therapy should go on for further study after the determination of the MLD. Bear in mind that the over-riding philosophical principle at work is that *the worst error one can make is to remove from further study a treatment that has a reasonable chance of being a successful therapy.* To put it another way, the researcher want to make very small the probability that a treatment that has a decent chance to be a successful therapy is rejected from further study.

There are two key phrases in the last sentence that drive the design: "very small" and "decent chance". Statisticians often view the number .05 as "very small". Keeping in mind that we are seeking therapies for dread diseases, we don't want "decent chance" to be too restrictive. So let's agree for now that a treatment has a decent chance to be a successful therapy if the probability for therapeutic benefit on a patient is 0.2.

Ed Gehan's design is to identify a certain number of patients (you will determine this number, so for now call it n), and administer the MLD to each one in succession. If the treatment is beneficial to patient 1, then the study terminates and the treatment goes on for further study. If the treatment fails on patient 1, then the MLD is administered to patient 2 (counterintuitive!!!). If the treatment is beneficial to patient 2, then the study terminates and the treatment goes on for further study. The study proceeds in this way until either it is successful on one of patients $1, 2, \ldots, n$ (in which case it is deemed worthy of further study) or it fails on all n and is rejected.

So the question is: Assuming that the responses (positive/negative) to the treatment on successive patients are independent events, what is the fewest number of patients needed to conduct the study? In other words, recalling the parameters established above, what is the fewest number of patients needed in order to make the probability $<.05$ that a drug that is 20% effective (i.e. will elicit positive response on 20% of patients) is rejected from further study.

Exercise 20. *What is the probability that a drug that is 20% effective fails on patient 1. On patients 1 and 2? On patients 1,2, and 3??*

Exercise 21. *What is the smallest value of n for which the probability of failure on all n patients of a treatment that is 20% effective is $<.05$?*

Exercise 22. *Suppose that we are more conservative in our measure of effectiveness, e.g. we will reject a drug only if it is unlikely to effective on 30% of patients. What happens to* n*?*

Exercise 23. *Suppose that we are less conservative in our measure of effectiveness. What happens to* n*?*

Exercise 24. *Interpret the meaning and consequences for* n *of different values for "very small".*