SURFACES

Surfaces are all that we see around us, so it is natural to try to understand all of them. That may seem overly ambitious, but it is possible provided we are careful in what we mean by "understand". For example, it is clear that a triangle and square are different polygons, but they have some properties in common - both are one dimensional, in the sense that a bug walking along them can only walk in one direction (forward or backward) except at the corners (which we call vertices), they determine a closed path, and separate the plane into two parts - the inside and outside. Most importantly for us, if they were made from a soft wire, each could be deformed into the other without cutting and soldering.

Topology is the branch of mathematics that deals with properties of objects that remain the same under smooth deformations. Two objects which can be smoothly deformed to each other, like our triangle and square, in fact any polygon and circle, are said to be topologically equivalent. They may be different geometrically (i.e. if we consider lengths and angles), but the same topologically. It is in the topological sense that we can obtain a classification of all surfaces.

A surface is a two dimensional object. This time our bug can crawl along the surface in two independent directions at most points. In principle a surface can be built up by gluing little square panels together. The squares might have to be very little and you may need lots of them to build a surface that has no visible corners, but from a topological point of view corners can be smoothed out. For instance,

Exercise 1. *Build a surface that is topologically equivalent to a sphere by gluing 6 identical square paper panels together.*

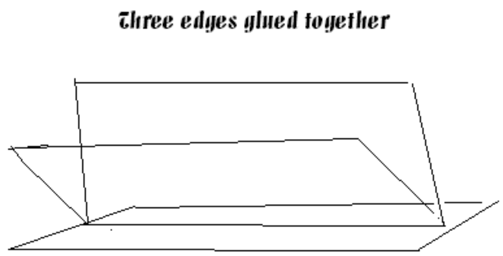
Exercise 2. *Try to build a sphere with identical paper panels of a different polygonal shape using fewer than six panels.*

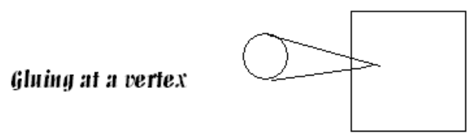
Our technical definition of a surface is any object that can be constructed by gluing together square paper panels according to two rules:

Rule 1: Panels can only be glued along edges

Rule 2: At most two edges can be glued together.

These rules eliminate surfaces like:

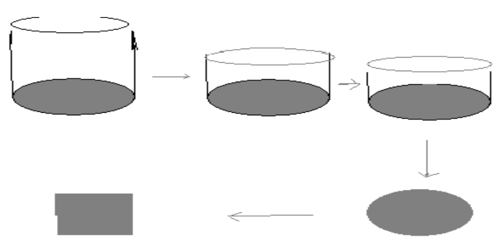




that can be analyzed by separating the surface into parts.

Look at some familiar surfaces. First of all we have a panel itself. Notice that size doesn't matter - a panel 100 miles by 100 miles is topologically equivalent to a single point.

A cylinder with an open top is topologically equivalent to a panel, hence to a point.

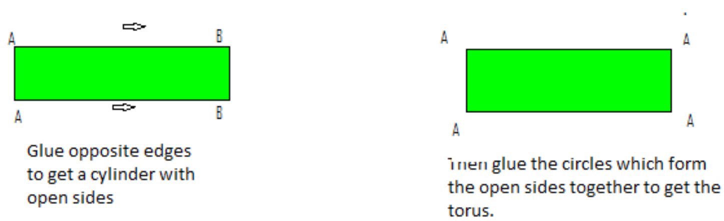


Notice that we began with a surface that doesn't lie in a plane but found a topologically equivalent surface that does lie in a plane. How about a cylinder with neither top nor bottom?

Exercise 3. *First find a surface lying in the plane that is topologically equivalent to a cylinder with neither top nor bottom. Then construct this "planar" surface from panels.*

The surface in the previous exercise is called an annulus (note the similarity with the Spanish word for ring - anillo).

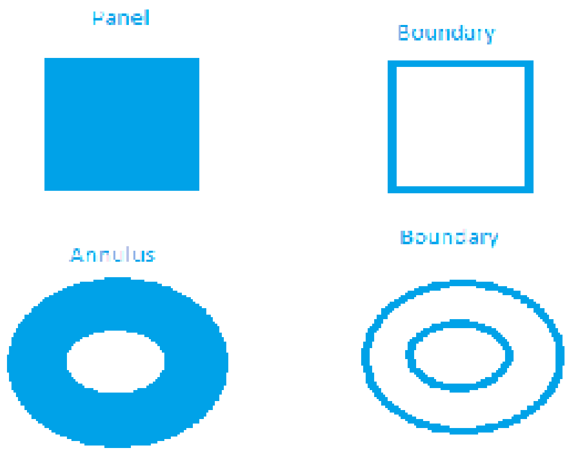
Mathematicians refer to the surface of a doughnut as a torus. The torus is a very interesting surface. Notice (see the animation on Wikipedia's entry on topology) that it is topologically equivalent to a coffee mug. Topologists are mathematicians who don't know their coffee mug from their doughnut. We can construct one from a single panel by gluing together the opposite sides as indicated by the arrows :



The Boundary

We are beginning to compile a collection of surfaces. The issue is to see which ones are different topologically and why. One place to start is at the boundary. It's intuitively clear that a panel and an annulus have a boundary, but a sphere and a torus don't. To be precise, consider a surface that has been constructed from paper panels. An edge in a panel is called *free* if it is not glued to any other edge.

The collection of free edges is called the *boundary* of the surface. Notice that the boundary of the panel is one curve and the boundary of the annulus consists of two separate curves.

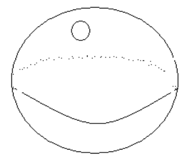


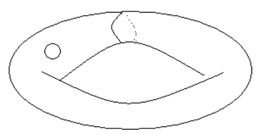
Exercise 4. *Construct a surface with three curves in the boundary. How would you construct one with n curves in the boundary?*

The number of curves in the boundary is a topological invariant of the surface, i.e. this number will be the same for all topologically equivalent surfaces. A surface is called **closed** if it has no free edges. This enables us to distinguish the panel and annulus from the torus and from each other.

How about distinguishing the torus from the sphere? Here are two ideas:

(1) What surfaces do you get if you puncture them? A punctured sphere is a topological disc. Is a tire with a nail puncture a topological disc?
(2) Imagine any circle on the sphere. It is the boundary of some circular patch lying on the sphere. (You can think of the equator as the boundary of either the northern or southern hemisphere. In the same way that a panel in the plane is contractible to a point, that circular patch is contractible to a point on the sphere. Can you find a circle on the torus that bounds no contractible patch?





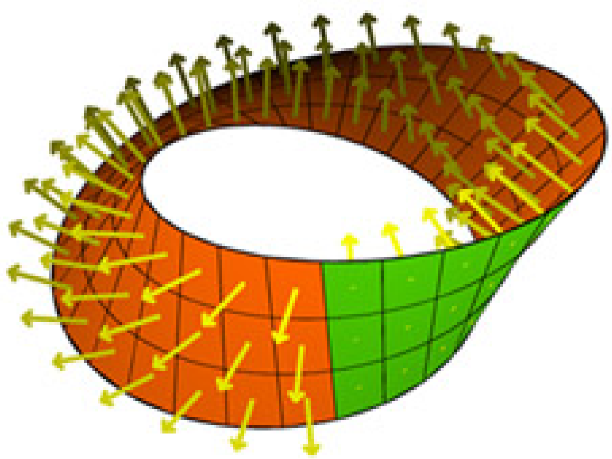
The idea of studying contractible patches leads to a branch of topology called homotopy theory that was first developed by the French mathematician Poincaré in the late 19^{th} and early 20^{th} centuries. We'll use different methodology to tell these surfaces apart.

Orientability

The Möbius band is constructed by gluing an edge of a panel to the opposite edge but with a half twist. Schematically:

How many curves in the boundary does it have? Is the Möbius band topologically equivalent to any of our other surfaces?

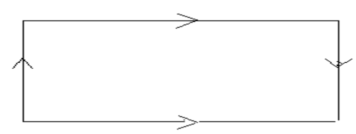
The Möbius band has no thickness. Imagine that we take a walk on it, starting at any point, walking parallel to the edge.



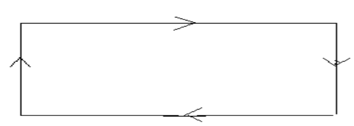
Thinking of the arrow points as our heads, what will you find when you return to your starting point the first time? We say that the Möbius band is nonorientable (because the path you traversed changes your right/left orientation). **A surface is nonorientable if there is no way to define right and left because of the existence of a path that returns you to the same point but with a different orientation. Otherwise the surface is called orientable.**

Here are two very famous surfaces that can be constructed from a single panel (but don't exist in our 3 dimensional spatial world):

(1) The Klein bottle (named for Felix Klein): Glue the opposite edges of a single panel together first to form a Möbius band then glue the remaining free edges together without a twist:



(2) The projective plane: Glue both pairs of opposite edges together with half twists:



Exercise 5. *Are these surfaces orientable?*

Let's make a table of what we know so far:

	# Curves in the Boundary	Orientable	Planar
Panel	1	Yes	Yes
Annulus	2	Yes	Yes
Möbius Band	1	No	No
Sphere	0	Yes	No
Torus	0		
Klein Bottle	0	No	No
Projective Plane	0		

Euler Characteristic

Some of our surfaces can't be distinguished so far. The great Swiss mathematician Leonhard Euler (pronounced like "oiler"), whose 300^{th} birthday was celebrated in 2007, comes to the rescue. The **Euler Characteristic** of a surface S, written $\chi(S)$ is the integer

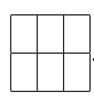
$$\#\ vertices - \#\ edges + \#\ panels$$

in any construction of the surface. Note that this definition is a bit fuzzy. There are many ways to construct a surface. Why should they all result in the same value of χ? They do, and this needs to be proved, but some experimentation makes the statement plausible.

Exercise 6. *Calculate $\chi(S)$ for each of the surfaces in the table in a couple of ways.*

$$\begin{aligned}\chi(Panel) &= \\ \chi(Annulus) &= \\ \chi(\text{Möbius}) &= \\ \chi(Sphere) &= \end{aligned}$$

For example a panel can be thought of as a rectangular region with 4 vertices, 4 edges and one panel, so $\chi(Panel) = 4 - 4 + 1 = 1$. But it's topologically equivalent to 6 panels glued together like this:



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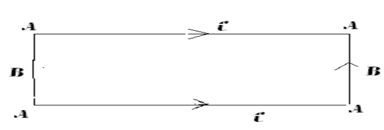
Here we have 12 vertices, 17 edges, and 6 panels, so $\chi(Panel) = 12 - 17 + 6 = 1$.

For the torus, Klein bottle and projective plane, it is useful to use the construction from a single panel. Recall that the torus is



So there is only one vertex, labelled A, two edges, labelled B and C, and only one panel.

Exercise 7. *Label the vertices and edges of the Klein bottle and projective plane.*

Exercise 8. *Fill in the following table*

$$\begin{aligned}\chi(Torus) &= 1-2+1=0\\ \chi(\mathrm{Pr}\,ojective\ Plane) &= \\ \chi(Klein\ Bottle) &= \end{aligned}$$

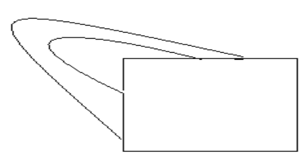
Note that properties 1,2, and 5 are sufficient to distinguish all of the surfaces in our catalog. In fact that is a Theorem:

Theorem 1. *A surface is completely determined up to topological equivalence by the number of curves in the boundary, whether or not it is orientable, and its Euler Characteristic.*

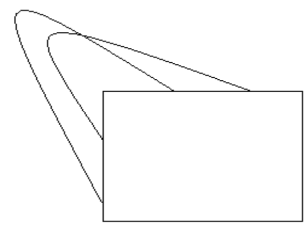
You can get some idea for why this is true by examining certain construction techniques. Here is another theorem:

Theorem 2. *Any surface can be constructed from a panel by applying some sequence of the following 4 constructions:*

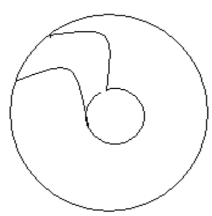
(1) Adding an EAR: This can only be done to a surface with at least one boundary curve. Glue two edges of a new panel to the same curve in the boundary. For example a panel with an ear is an annulus.



(2) Adding a TWISTED EAR: This can only be done to a surface with at least one boundary curve. Glue two edges of a anew panel to the same curve in the boundary, but with a half twist. What is the resulting surface?



(3) Adding a BRIDGE: This can only be done to a surface with at least two curves in the boundary. Glue two edges of a new panel to different curves in the boundary. What do you get if you add a bridge to an annulus?



The resulting surface, a bridged annulus, can be viewed in another way. Stretch the free edges of the bridge until they almost meet, leaving a slit. Now stretch the base to almost close the slit. What we have is a punctured torus.

(4) Adding a LID This can only be done to a surface with at least one curve in the boundary: Glue all 4 edges of a new panel along the entire length of a boundary curve.

Exercise 9. *Reverse the equivalence above of a panel with a cylinder with an open top. Now which of our surfaces do we get if we add a lid to a panel?*

We can derive some equations:

$$\begin{aligned} panel + ear &= annulus. \\ panel + twisted\ ear &= ? \\ annulus + bridge + lid &= ? \end{aligned}$$

You might ask why we didn't include a twisted bridge (i.e. gluing two edges of a new panel to different curves in the boundary but with a half twist) . This can be accomplished by adding two twisted ears and a lid. The easiest way to see this is by adding a twisted bridge to an annulus and then puncturing the resulting surface on the twisted bridge. Stretch the puncture into a slit and slide the bases of the two halves of the bridge to different curves in the boundary of the annulus. These become the two twisted ears.

Let's examine the affect of these 4 operations on our three important topological invariants:

	# Curves in the Boundary	Orientable	χ
Adding an Ear	+1	No Change	-1
Adding a Twisted Ear	No change	Orientable→ Nonorientable	-1
Adding a Bridge	-1	No Change	
Adding a Lid			

For example, what does our information tell us about how to construct the Klein bottle? The Klein bottle has no curves in the boundary, is non orientable, and has Euler Characteristic equal to 0. Start with a panel. Add a twisted ear to get a nonorientable surface with Euler characteristic 0 and one curve in the boundary. Add another twisted ear to get a nonorientable surface with Euler characteristic -1 and one curve in the boundary. Now add a lid to increase the Euler characteristic to 0 and reduce the number of boundary curves to 0. Since the surface remains nonorientable, it must be the Klein bottle.

Exercise 10. *How would you build the projective plane?*

The culmination of this topological theory of surfaces are two theorems.

Theorem 3. *A closed orientable surface is either a sphere or a torus with one or more holes.*

You can construct a torus with two holes by gluing two punctured tori together along the punctures. But how would you construct it with our construction techniques starting from a panel? Remember how we built a torus by adding an ear, a bridge, and a lid. For the two holed torus add two ears, then two bridges. How many lids do we need to get a closed surface? What is the Euler characteristic of the sculpture in the entry of library? How do you construct a torus with many holes and what is its Euler characteristic? .

For nonorientable surfaces we have the:

Theorem 4. *A closed nonorientable surface is either a projective plane or obtained by gluing together one or more projective planes along punctures.*

In particular, the Klein bottle is obtained by gluing 2 punctured projective planes - this is essentially the construction above as two Möbius bands plus a lid.

One important consequence of the first of these theorems is that the sphere is the only closed surface on which all curves are contractible. It was conjectured by Poincaré at the end of the 19^{th} century that an analogous theorem was true in all dimensions. It was proved true in all dimensions ≥ 5 in the 1960's, in dimension 4 in the 1980's (I was in the room when it was announced), and in dimension 3, amidst some controversy, only within the last 10 years. All three of the mathematicians who solved the parts of Poincare's conjecture were awarded the Field's medal. The most recent one refused his.