Exercise 2.28. Show that compactness is a topological property and give examples to show that closedness and boundedness are not.

Compactness is an important property and is the topologists' form of finiteness, in the sense that a compact set does not go on forever. One's first impression is that a bounded set should be "finite," but we have seen that the interval $(-1, 1)$ is topologically equivalent to $(-\infty, \infty)$. One way of thinking about this is to imagine walking along the interval $(-1, 1)$, towards 1, but for some reason (an increase in gravity, or the unnerving fact that one's legs seem to be getting shorter and shorter), the closer one gets to 1, the smaller steps one has to take, so that one never reaches 1. In this sense, $(-1, 1)$ is endless. On the other hand, $[-1, 1]$ is compact and does not go on forever, since it has ends. No topological property should be based solely on distance, as is boundedness, since in topology distance means very little.

2.5 Connected sets

Another fundamental notion is the number of pieces, or *components*, an object has. If one thinks about what "connected" means, an object X is connected, or has only one piece, if all its parts are stuck to each other. Now take the idea "stuck to each other" and look at Figure 2.18.

Fig. 2.18. X is connected, but Y is not

X and Y each have been divided into two parts, where B looks exactly like B', and A looks like A' with the point $\mathbf{x}$ added on. The point $\mathbf{x}$ sort of glues A and B together to make X connected, and, being absent from Y, is the gap where Y falls apart into two pieces. Note that $\mathbf{x}$ is in the set A and is a limit point of both A and B, so that $\mathbf{x}$ is infinitely close to the set B; in a sense, $\mathbf{x}$, and so A, cannot be separated from B. This is how the formal definition of connectedness is derived.

(2.27) Definition. *A set S is connected if whenever S is divided into two non-empty disjoint sets, so that*

$$S = A \cup B,\ A \neq \emptyset,\ B \neq \emptyset,\ A \cap B = \emptyset$$

then either A or B contains a limit point of the other.