FUNDAMENTAL MATHEMATICS

1) Determine whether the following sets are basis of $\mathbb{R}^3$ or not.
i) $\{(2,1,-2),(-2,-1,2),(2,4,-4)\}$, ii) $\{(1,1,1),(1,5,6),(6,2,1)\}$.

2) Let V be the following subset of $\mathbb{R}^4$

$$V = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 | x_1 - x_2 + x_3 - x_4 = 0\}$$

Prove that V is a subspace of $\mathbb{R}^4$. Find a basis of V.

3) Let V_1 and V_2 be vector spaces. We let $V_1 \oplus V_2$ be the vector space, which as a set is the set of all pairs (v_1, v_2), where $v_1 \in V_1$ and $v_2 \in V_2$. The addition and multiplication on a scalar is defined by

$$(v_1, v_2) + (v_1', v_2') = (v_1 + v_1', v_2 + v_2'),$$
$$\lambda(v_1, v_2) = (\lambda v_1, \lambda v_2).$$

Prove that $\mathbf{dim}(V_1 \oplus V_2) = \mathbf{dim}(V_1) + \mathbf{dim}(V_2)$.

Hint. You can take bases of V_1 and V_2 and try to construct a basis of $V_1 \oplus V_2$. Alternatively, you can apply the rank-nullity theorem to the linear map $p : V_1 \oplus V_2 \to V_2$, given by $p(v_1, v_2) = v_2$.

4) Let U and W be subspaces of a vector space V. Prove that

$$\mathbf{dim}(U + W) = \mathbf{dim}(U) + \mathbf{dim}(W) - \mathbf{dim}(U \cap W).$$

Hint. Consider the linear map $f : U \oplus W \to V$, given by $f(u, w) = u - w$, where $U \oplus W$ is the same as in the previous problem. Now use the result of the previous problem and the rank-nullity theorem applied on f.

5) Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear map given by

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 - x_2, x_2 + x_3)$$

Find the matrix corresponding to f in the basis $\{(1,1,0),(-1,1,0),(1,1,1)\}$.