# UNIVERSITY OF NEWCASTLE UPON TYNE

## SCHOOL OF MATHEMATICS & STATISTICS

**SEMESTER 2**
2002-2003

**MAS 252/652**

**Mathematical Modelling**

**Time : 1 hour 30 minutes**

*Credit will be given for ALL answers to questions in Section A and the best TWO answers to questions in Section B.*

*No credit will be given for other answers, and students are strongly advised not to spend time producing answers for which they will receive no credit.*

*There are FIVE questions in Section A and THREE questions in Section B. Marks allocated to each question are indicated. However, you are advised that marks indicate the relative weight of individual questions; they do not correspond directly to marks on the University scale.*

## SECTION A

**A1.** Find and classify all the singular points (excluding the point at infinity) of the equation

$$2x^2(1+x)\frac{d^2y}{dx^2} - 3(1-x)\frac{dy}{dx} + 2y = 0.$$

**[8 marks]**

**A2.** When the method of Frobenius (with index $\lambda$) is applied to a particular differential equation which contains the parameter $k$, the indicial equation is found to be

$$\lambda^2 = k \quad (\text{with } a_0 \neq 0);$$

the equation involving the coefficient $a_1$ (in the usual notation) is

$$a_1\left\{(\lambda+1)^2 - k\right\} = 0$$

and the recurrence relation is

$$a_m\left\{(\lambda+m)^2 - k\right\} = a_{m-2}, \qquad m = 2, 3, \ldots .$$

In the cases

**(a)** $k = 2$; **(b)** $k = 1$; **(c)** $k = 0$,

state the structure of the general solution, paying particular attention to whether *log* terms appear.

**[10 marks]**

**A3.** Find all the eigenvalues ($\lambda$), and the corresponding eigenfunctions, for the problem described by

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + \lambda y = 0, \quad 0 \le x \le 1,$$

with $y(0) = y'(1) - 3y(1) = 0$, where the prime denotes the derivative with respect to $x$. (It is sufficient to obtain the relevant non-zero solution.)

**[8 marks]**

**A4.** Find the general solution of the equation

$$2y\frac{\partial u}{\partial x} + 3x^2\frac{\partial u}{\partial y} = -2yu,$$

by introducing the characteristic variable $\xi = y^2 - x^3$. Hence find that solution which satisfies : $u = \sin(y^2)$ on $x = 0$.

**[8 marks]**

**A5.** Show that the partial differential equation

$$x^2\frac{\partial^2 u}{\partial x^2} + 4xy\frac{\partial^2 u}{\partial x \partial y} + (3y^2 + y)\frac{\partial^2 u}{\partial y^2} = u + \frac{\partial u}{\partial x}$$

is of elliptic type for $0 < y < 1$ and all $x \ne 0$; what is it for $y > 1$ and $y < 0$ (and all $x \ne 0$)?

**[6 marks]**

## SECTION B

**B6.** Use the method of Frobenius to find a power-series representation, about $x = 0$, of the general solution of the equation

$$9x^2(1+x)\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} - y = 0.$$

Your solution should include the indicial equation and the recurrence relation; you need retain no terms beyond $x^{10/3}$.

**[30 marks]**

**B7.** A particular heat conduction problem is described by the equation

$$\frac{\partial u}{\partial t}=\frac{1}{4}\frac{\partial^2 u}{\partial x^2}+ku,$$

where $k$ is a constant.

**(a)** Use the method of separation of variables to find the most general solution which satisfies the boundary conditions

$$u(0,t)=u(\ell,t)=0 \quad (t\geq 0);$$

you may find it convenient to associate the term in $k$ with the $t$ behaviour. Deduce that your solution satisfies

$$u(x,t)\to 0 \text{ as } t\to\infty$$

if $k<\pi^2/(4l^2)$.

**[15 marks]**

**(b)** Now seek a solution of the equation in the form

$$u(x,t)=e^{kt}f(xt^n)$$

and hence find the value of $n$. Obtain the general solution for $u$ which satisfies

$$u\to 0 \text{ as } t\to 0 \ (x>0)$$

and

$$e^{-kt}u\to 1 \text{ as } t\to\infty \ (x>0).$$

**[15 marks]**

**B8.** Show that the partial differential equation

$$4y^2\frac{\partial^2 u}{\partial x^2}+4y\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=4y^2\frac{\partial u}{\partial x}+\left(2y+\frac{1}{y}\right)\frac{\partial u}{\partial y}$$

is of parabolic type (for all $y \neq 0$). Solve the equation defining the characteristics, introduce an appropriate pair of new variables and hence write the equation in a canonical form. Give the general solution of this equation.

**[30 marks]**