Ito's Lemma

Recall that a Wiener process dW is normalized to have expectation $E(dW^2) = dt$.

(1)

A specialized form of Ito's lemma can be found for instance on web page

http://en.wikipedia.org/wiki/It%C5%8D's_lemma

If we take the first two equations from this web page and change variables to those of the assignment,

$$dS = a(S,t)dt + b(S,t)dX, \tag{1.1}$$

and collect together the terms containing $\partial f/\partial S$, we obtain exactly the requested statement

$$df = \frac{\partial f}{\partial S}dS + \left(\frac{\partial f}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 f}{\partial S^2}\right)dt. \tag{1.2}$$

An informal proof of equation (1.2) is further given on that web page. The key element of the informal proof, in our notations here, is the replacement

$$dX^2 \to E(dX^2) = dt, \tag{1.3}$$

in the limit $dt \to 0$, whose proof is too involved to be explained in a basic text.

The rest of the informal proof there consists of simple algebraic book-keeping. The same book-keeping is done in our notations here in the next question.

(2)

If

$$dS = \mu S dt + \sigma S dX, \tag{2.1}$$

$\mu \geq 0$, $\sigma > 0$, dX is a Wiener process, $P_m > 0$, and

$$\xi = \frac{S}{S + P_m} = 1 - \frac{P_m}{S + P_m}, \tag{2.2}$$

we use Taylor expansion and obtain

$$d\xi = \frac{P_m}{(S+P_m)^2}dS - \frac{P_m}{(S+P_m)^3}dS^2 + O(dS^3) =$$

$$= \frac{(1-\xi)^2}{P_m}dS - \frac{(1-\xi)^3}{P_m^2}dS^2 + O(dS^3). \tag{2.3}$$

In the same way as it is done for question (1) in the cited web page, we expand

$$dS^2 = \mu^2 S^2 dt^2 + 2\mu\sigma S^2 dt dX + \sigma^2 S^2 dX^2, \tag{2.4}$$

make replacement (1.3) and note that the remaining terms in equation (2.2) are of higher order than dt and so can be dropped, so that

$$dS^2 = \sigma^2 S^2 dt + o(dt). \tag{2.5}$$

Using equations (2.1) and (2.5) and retaining only the expansion terms up to the order of dt in equation (2.3), we obtain

$$d\xi = \frac{(1-\xi)^2}{P_m}(\mu S dt + \sigma S dX) - \frac{(1-\xi)^3}{P_m^2}\sigma^2 S^2 dt. \tag{2.6}$$

Noting from equation (2.2) that

$$S = \frac{P_m \xi}{1-\xi} \tag{2.7}$$

and substituting equation (2.7) into equation (2.6), we obtain

$$\begin{aligned} d\xi &= (1-\xi)\xi(\mu dt + \sigma dX) - \sigma^2(1-\xi)\xi^2 dt = \\ &= \xi(1-\xi)(\mu - \sigma^2\xi)dt + \sigma\xi(1-\xi)dX \\ &= a(\xi)dt + b(\xi)dX, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} a(\xi) &= \xi(1-\xi)(\mu - \sigma^2\xi), \\ b(\xi) &= \sigma\xi(1-\xi). \end{aligned} \tag{2.9}$$

From equation (2.9) we see that $a(\xi)$ and $b(\xi)$ do indeed have the properties

$$a(0) = a(1) = b(0) = b(1) = 0. \tag{2.10}$$