Matrix Theory – Homework 7

Prove the following in several stages:
The composition of any two reflections, whose lines of reflection are orthogonal, is a half-turn.

We will work in the vector space $\mathbb{R}^2$.

First Stage
Refer to the diagram provided. The line l makes some angle θ with the x-axis.
Let us suppose that the point $(c,d)\in l$. (It is clear that, if we were given θ , we could find such a point (c,d) , and if we were given (c,d) , we could find θ .)
Informally, we define $f:\mathbb{R}^2\rightarrow\mathbb{R}^2$ to be the action of reflecting any point $\vec{x}$ through l .
Now consider any arbitrary $\vec{x}=(x_{1,}x_2)$. (Ignore $(e_{1,}e_2)$ in the diagram for now.)

1. Using the notation of projections, derive an expression for the vector $f(\vec{x})$.

You may suppose that $\vec{c}$ is the vector starting at the origin and going to (c,d) .
Do **not** attempt to evaluate it further at this stage; just leave things in terms of $\vec{c}$ and $\vec{x}$.

Second Stage
1. Evaluate your expression further, using the formula for the projection of a point. Your final expression must be given in the form $A\vec{x}$, for some matrix *A* whose entries are in terms of *c* and *d*.

Notice that you have constructed a formula for the action of reflecting $\mathbb{R}^2$ through the line going through the origin and (c,d) , for **any** $c,d\in\mathbb{R}$.

Third Stage (sanity check Using your formula, let $k\in\mathbb{R}$ and evaluate $f((kc,kd))$.

1. What d you get?

Fourth Stage
Consider a specific situation and draw a diagram of it: $\vec{x}=(2,0)$, $(c,d)=(3,\sqrt{(3)})$.
Take $(e_{1,}e_2)$ to be the point of intersection of l and the line orthogonal to l intersecting $\vec{x}$.

1. What is θ ?

2. What is $(e_{1,}e_2)$?

You should notice something interesting about the triangle $\Delta(0,0),(2,0),f((2,0))$.

3 Figure out $f((2,0))$ using standard geometry and this observation.

4. Now apply your formula to determine $f((2,0))$. Do you get the same answer?

Fifth Stage
Now rotate your point (c,d) ninety degrees counterclockwise.

1. What coordinates does it have?

2. Define the line m to be the line through this point and the origin. Define $g:\mathbb{R}^2\rightarrow\mathbb{R}^2$ to be the action of reflecting points through m . So $g(\vec{x})=B\vec{x}$, for some matrix *B*.

3. According to your formula from the Second Stage, what is B?

Sixth Stage

Now we note that $g(f(\vec{x}))=g(A\vec{x})=B(A\vec{x})=(BA)\vec{x}$.

1. What is BA? Is this the matrix for the half-turn linear operator?