approximately $M^3/3$ floating point operations are required to solve a system of M linear equations. The Thomas algorithm is therefore very efficient, especially for large systems.

When the heat equation is being solved using an implicit difference scheme, a linear system of the form $AU_{n+1} = \mathbf{d}_n$ must be solved at every time step. The matrix A is unchanged between time steps; only the right-hand side $\mathbf{d}_n$ depends on the solution at time t_n and has to be re-computed. Further savings may then be attained in the Thomas algorithm by computing the multipliers m_i and the elements b_i' once and for all before starting the time steps; m_i can replace a_i. At each time step only d_i, d_i' and U_{n+1} are computed, saving a further $2M$ floating point operations per time step.

4.2 Three-Level Difference Schemes

The Crank-Nicholson method attains second order accuracy in both space and time by means of central differences. This improvement has to be balanced against the extra complication involved in the use of an implicit method. If more than two time levels are used in deriving a difference scheme, we may improve the accuracy while retaining the efficiency of an explicit method.

Using central differences for both time and space derivatives in the heat equation $u_t = au_{xx}$ leads to the explicit three-level scheme (known as *Richardson's scheme*)

$$\frac{1}{2k}\left(U_i^{n+1} - U_i^{n-1}\right) = \frac{a}{h^2}\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right).$$

The local truncation error of this scheme is of order $O(k^2) + O(h^2)$ and the amplification factor g satisfies the quadratic equation

$$g^2 + 8gr\sin^2(\theta/2) - 1 = 0.$$

This equation has two unequal real roots and their product is -1; hence the magnitude of one of the roots is greater than unity. The scheme is therefore unstable for all r (i.e. it is unconditionally unstable) and hence of no use in practice.

A three-level scheme which is explicit and unconditionally stable may be obtained by a slight modification of Richardson's scheme:

$$\frac{1}{2k}\left(U_i^{n+1} - U_i^{n-1}\right) = \frac{a}{h^2}\left(U_{i-1}^n - U_i^{n-1} - U_i^{n+1} + U_{i+1}^n\right)$$

or

$$U_i^{n+1} = \frac{2r}{1+2r}\left(U_{i-1}^n + U_{i+1}^n\right) + \frac{1-2r}{1+2r}U_i^{n-1}.$$

This scheme (the scheme of *Du Fort and Frankel*) has an amplification factor g which satisfies the equation

$$(2r+1)g^2 - 4rg\cos\theta - (1-2r) = 0$$