Two Hours

UNIVERSITY OF MANCHESTER

Random Processes MT2242

Tuesday 31st May, 2005
9.45a.m. - 11.45a.m.

Electronic calculators may be used, provided that they cannot store text.

Answer **ALL** four questions in **SECTION A** (40 marks in all)
and
TWO of the three questions in **SECTION B** (20 marks each)
The total number of marks on the paper is 80. A further 20 marks are available from work during the semester making a total of 100.

SECTION A

Answer **ALL** four questions

A1. (i) Let X be a random variable with probability mass function

$$f_X(n) = P(X = n) = e^{-6}\frac{6^n}{n!}, n \geq 0.$$

Determine the probability generating function of X. [5 marks]

(ii) Suppose that the probability generating function of a random variable X is given by

$$G_X(s) = s^2 \frac{\frac{1}{3}}{1 - \frac{2}{3}s}$$

Determine the probability mass function of X. [5 marks]

(Hint: You may use the formula: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, 0 \leq x < 1$.)

A2. Let $Z_n, n \geq 0$ be a branching process with offspring distribution

$$P(N = 0) = \frac{1}{4}, P(N = 1) = \frac{1}{2}, P(N = 2) = \frac{1}{4}.$$

(i) Find the offspring generating function $G(s)$ and the expectation of N. [4 marks]

(ii) Determine the probability generating function of Z_2 and write down $P(Z_2 = 3)$.

[6 marks]

A3. Let $(S_n, n \geq 0)$ be a random walk starting from zero, i.e., for $n \geq 1$, $S_n = \sum_{i=1}^{n} Y_i$, where $Y_1, Y_2, ...$ are independent with $P(Y_i = 1) = p, P(Y_i = -1) = q, p + q = 1$.

(i) Compute $P(S_1 = 1, S_3 = 1, S_4 = 2)$. [5 marks]

(ii) Write down the recurrence probability f^*. State the definition of a random walk being transient. For what values of p, is the random walk transient? [5 marks]

A4. Let $(N_t, t \geq 0)$ be a Poisson process of rate λ.

(i) State the distribution of N_t and the distribution of $N_t - N_s$ for $0 < s < t$. [4 marks]

(ii) Calculate $E[N_t N_s]$ for $0 \leq s < t$. [6 marks]

SECTION B

Answer **TWO** of the three questions

B5. In the spreading of a rumour, each person who is told the rumour has probability $\frac{2}{3}$ of believing it and only those who believe it will pass the rumour on. Suppose that each person hearing and believing the story on the n^{th} day will pass it on to two further people on the $(n+1)^{th}$ day. Assume that all individuals behave independently of each other and the story is never told to someone who has already heard it.

If the rumour originates on day 1 from a single individual telling the story to two friends, let Z_n denote , for $n \geq 1$, the number of people who are told the story and believe (!!) it on the n^{th} day, and $Z_0 = 1$.

(i) Explain that $(Z_n, n \geq 0)$ is a branching process. [3 marks]

(ii) Find the offspring probability mass function and the offspring generating function $G(s)$. [7 marks]

(iii) What is the probability that the spread of the rumour ultimately ceases? [10 marks]

B6. The Gambler's Ruin Problem. Soppose that a gambler has an initial capital k units and the bank has an initial capital m units. The gambler makes a simple bet, over and over again, and on each bet he has probability p of winning 1 unit from the bank, and probability $q = 1 - p$ of losing 1 unit to the bank. The gambler continues betting till (i) he loses all his money (he is ruined); or (ii) the bank is ruined (the gambler's capital increases to $l = m+k$ units). Let S_n denote the fortune of the gambler after n-th bet.The $\{S_n, n \geq 0\}$ is a simple random walk. Let r_j denote the probability that the gambler ruins the bank given the gambler has j units.

(i) Explain why $r_k = P_k(min(n; S_n = l) < min(n; S_n = 0))$. [2 marks]

(ii) Write down the values of r_0 and r_l. [3 marks]

(iii) By considering the possible values of S_1, show that for $0 < j < l$,

$$r_j = pr_{j+1} + qr_{j-1}.$$

[5 marks]

(iv) Assume $p \neq q$. Deduce from (i) and (ii) that

$$r_k = \frac{1 - (\frac{q}{p})^k}{1 - (\frac{q}{p})^l}.$$

[10 marks]

B7. The number N_t of road traffic accidents in $(0, t]$ in a city forms a Poisson process of rate λ. The number X of people seriously injured in each accident may be assumed to be random variables, independent of each other and of the Poisson process N_t, and each taking the value $n = 0, 1, 2, ...$ with probability pq^n, $q = 1 - p$. Let $Y(t)$ be the number of people seriously injured in road traffic accidents in $(0, t]$.

(i) Express $Y(t)$ as a random sum. [6 marks]

(ii) Find the probability generating functions of N_t and X. [8 marks]

(iii) Find the probability generating function of $Y(t)$, $P(Y(t) = 0)$ and $E[Y(t)]$.

[6 marks]

(You may use the Random Sums Lemma)