structure $(\mathbf{R}, \mathbf{x}_1)$ and $(\mathbf{R}, \mathbf{x}_2)$ are distinct, they determ[illegible] diffeomorphic differentiable manifolds.

12. (*The orientable double covering*). Let $M^n$ be a non-orientab[illegible] differentiable manifold. For each $p \in M$, consider the se[illegible] of bases of $T_pM$ and say that two bases are equivalent if the[illegible] are related by a matrix with positive determinant. This is a[illegible] equivalence relation and separates $B$ into two disjoint sets. Le[illegible] $\mathcal{O}_p$ be the quotient space of $B$ with respect to this equivalen[illegible] relation. $O_p \in \mathcal{O}_p$ will be called an *orientation* of $T_pM$. Le[illegible] $\overline{M}$ be the set

$$\overline{M} = \{(p, O_p); p \in M, O_p \in \mathcal{O}_p\}.$$

Let $\{(U_\alpha, \mathbf{x}_\alpha)\}$ be a maximal differentiable structure on $M$ [illegible] and define $\overline{\mathbf{x}}_\alpha : U_\alpha \to \overline{M}$ by

$$\overline{\mathbf{x}}_\alpha(u_1^\alpha, \ldots, u_n^\alpha) = (\mathbf{x}_\alpha(u_1^\alpha, \ldots, u_n^\alpha), [\frac{\partial}{\partial u_1^\alpha}, \ldots, \frac{\partial}{\partial u_n^\alpha}]),$$

where $(u_1^\alpha, \ldots, u_n^\alpha) \in U_\alpha$ and $[\frac{\partial}{\partial u_1^\alpha}, \ldots, \frac{\partial}{\partial u_n^\alpha}]$ denotes the ele[illegible] ment of $\mathcal{O}_p$ determined by the basis $\left\{\frac{\partial}{\partial u_1^\alpha}, \ldots, \frac{\partial}{\partial u_n^\alpha}\right\}$. Prove that:

(a) $\{(U_\alpha, \overline{\mathbf{x}}_\alpha)\}$ is a differentiable structure on $\overline{M}$ and that the manifold $\overline{M}$ so obtained is orientable.

(b) The mapping $\pi : \overline{M} \to M$ given by $\pi(p, O_p) = p$ is differ[illegible] entiable and surjective. In addition, each $p \in M$ has a[illegible] neighborhood $U \subset M$ such that $\pi^{-1}(U) = V_1 \cup V_2$, where $V_1$ and $V_2$ are disjoint open sets in $\overline{M}$ and $\pi$ restricted to each $V_i$, $i = 1, 2$, is a diffeomorphism onto $U$. For this reason, $\overline{M}$ is called the *orientable double cover of* $M$.

(c) The sphere $S^2$ is the orientable double cover of $P^2(\mathbf{R})$ and the torus $T^2$ is the orientable double cover of the Klein bottle $K$.

CHAPTER 1

# RIEMANNIAN METRICS

## [illegible]ntroduction

[illegible]ically, Riemannian geometry was a natural development of the [illegible]ntial geometry of surfaces in $\mathbf{R}^3$. Given a surface $S \subset \mathbf{R}^3$, we [illegible] natural way of measuring the lengths of vectors tangent to [illegible]mely: the inner product $\langle v, w \rangle$ of two vectors tangent to $S$ at a [illegible] of $S$ is simply the inner product of these vectors in $\mathbf{R}^3$. The [illegible] compute the length of a curve is, by definition, to integrate [illegible]ength of its velocity vector. The definition of $\langle\ ,\ \rangle$ permits us [illegible]asure not only the lengths of curves in $S$ but also the area of [illegible]ns in $S$, as well as the angle between two curves, and all the [illegible] "metric" ideas used in geometry. More generally, these notions [illegible] to define on $S$ certain special curves, called geodesics, which [illegible] the following property: given any two points $p$ and $q$ on [illegible]desic, sufficiently close (in a sense to be made precise later, [illegible] Chap. 3), the length of such a curve is less than or equal to [illegible]ength of any other curve joining $p$ to $q$. Such curves behave, [illegible] situations, as if they were "the straight lines" of $S$, and, [illegible] shall see later, play an important role in the development of [illegible]metry.

Observe that the definition of the inner product at each point [illegible] $S$, yields, equivalently, a quadratic form $I_p$, called the first [illegible]amental form of $S$ at $p$, defined in the tangent plane $T_pS$ by [illegible] $= \langle v, v \rangle$, $v \in T_pS$.

The crucial point of this development was an observation [illegible] by Gauss in his famous work (see Gauss [Ga]) published in [illegible] In this work, Gauss defined a notion of curvature for surfaces, [illegible] measures the amount that $S$ deviates, at a point $p \in S$, from [illegible]gent plane at $p$. In modern notation, Gauss' definition can be [illegible]ssed in the following terms. Define a mapping $g: S \to S^2 \subset \mathbf{R}^3$ [illegible] into the unit sphere $S^2$ of $\mathbf{R}^3$, associating to every $p \in S$ a unit vector $N(p) \in S^2$ normal to $T_pS$; if $S$ were orientable then $g$ would be well-defined and differentiable on $S$. During Gauss' time, the notion of orientation of surfaces was not well-understood (in truth, it wasn't until 1865 that Möbius presented his famous example, well-known today as the Möbius band), and so $g$ was defined on "pieces" of $S$. In any case, $g$ is differentiable and it is possible then to speak of its differential $dg_p: T_pS \to T_{g(p)}S^2$. Since $N(p)$ is normal to $T_pS$, we can identify the two vector spaces $T_pS$ and $T_{g(p)}S^2$, and thus it makes sense to speak of the determinant of the linear map $dg_p$. Gauss defined his curvature as $K(p) = \det(dg_p)$ and showed that it agreed with the product of the principal curvatures introduced in 1760 by Euler.

Perhaps it is worthwhile mentioning that Euler defined the principal curvatures $k_1$ and $k_2$ of a surface $S$ by considering the curvature $k_n$ of curves obtained by intersecting $S$ with planes normal to $S$ at $p$ and taking $k_1 = \max k_n$ and $k_2 = \min k_n$. At the time of Gauss it was not at all clear that one function or the other of $k_1$ and $k_2$, would be an adequate definition of curvature. Gauss considered that the facts which he had obtained about $K$ justified the choice of $K = k_1k_2$ as the curvature of $S$.

The facts that Gauss alluded to were the following. In the first place, the curvature, as defined above, depends only on the manner of measuring in $S$, that is, only on the first fundamental form $I$. Secondly, the sum of the interior angles of a triangle formed by geodesics differs from 180° by an expression that depends only on the curvature and the area of the triangle.

Everything indicates that Gauss perceived very clearly the profound implications of his discovery. In fact, one of the fundamental problems during Gauss' time was to decide if the fifth postulate of Euclid ("Given a straight line and a point not on the line then there is a straight line through the point which does not meet the given line") was independent of the other postulates of geometry. Although without immediate applications, the question leads to philosophical implications of primary importance. Earlier, it had been established that Euclid's fifth postulate is equivalent to the fact that the sum of the interior angles of a triangle equals 180°. The discovery of Gauss implied, among other things, that it would be possible to imagine a geometry (at least in dimension two) that depended on a fundamental quadratic form given in an arbitrary



manner (without regard to the ambient space). In such a geometry, defining straight lines as geodesics, the sum of the interior angles of a triangle would depend on the curvature and, as Gauss actually verified, its difference from 180° would be equal to the integral of the curvature over the triangle. Gauss, however, did not have the necessary mathematical tools available to develop his ideas (what he lacked was essentially the idea of a differentiable manifold) and he preferred not to discuss this topic openly. The actual appearance of a non-euclidean geometry was due, independently, to Lobatchevski (1829) and Bolyai(1831).

The ideas of Gauss were taken up again by Riemann in 1854 (see Riemann [Ri]), even though he was still without an adequate definition of a manifold. Using intuitive language and without proof, Riemann introduced what we call today a differentiable manifold of dimension $n$. He further associated to every point of the manifold a fundamental quadratic form and then generalized the idea of Gaussian curvature to this situation (cf. Chap. 4). Furthermore, he stated many relations between the first fundamental quadratic form and the curvature that were only proved decades later. The reading of his work makes it clear that Riemann was motivated by the fundamental question implicit in the development of non-euclidean geometries, namely, the relationship between physics and geometry.

It is curious to observe that the concept of differentiable manifold, necessary for the formalization of the work of Riemann, only appeared explicitly in 1913 in the work of H. Weyl which made precise another of Riemann's audacious concepts, namely, Riemann surfaces. But that is another story.

Due to the lack of adequate tools, Riemannian geometry as such developed very slowly. An important outside source of stimulation was the application of these ideas to the theory of relativity in 1916. Another fundamental step was the introduction of the parallelism of Levi-Civita. We shall return to this topic in the next chapter. Our object here is not to write a complete history of Riemannian geometry but simply to trace its origin and supply motivation for what is to follow.

Our point of departure will be a differentiable manifold on which we introduce at each point a way of measuring the length of tangent vectors. This measurement should change differentiably from point to point. The explicit definition will be given in the next section.

For the remainder of this book, the differentiable manifolds considered will be assumed to be Hausdorff spaces with countable bases. "Differentiable" will signify "of class $C^\infty$", and when $M^n = M$ denotes a differentiable manifold, $n$ denotes the dimension of $M$.

## 2. Riemannian Metrics

2.1 DEFINITION. A *Riemannian metric* (or *Riemannian structure*) on a differentiable manifold $M$ is a correspondence which associates to each point $p$ of $M$ an inner product $\langle\ ,\ \rangle_p$ (that is, a symmetric, bilinear, positive-definite form) on the tangent space $T_pM$, which varies differentiably in the following sense: If $\mathbf{x}: U \subset \mathbf{R}^n \to M$ is a system of coordinates around $p$, with $\mathbf{x}(x_1, x_2, \ldots, x_n) = q \in \mathbf{x}(U)$ and $\frac{\partial}{\partial x_i}(q) = d\mathbf{x}_q(0, \ldots, 1, \ldots, 0)$, then $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q)\rangle_q = g_{ij}(x_1, \ldots, x_n)$ is a differentiable function on $U$.

It is clear this definition does not depend on the choice of coordinate system.

Another way to express the differentiability of the Riemannian metric is to say that for any pair of vector fields $X$ and $Y$, which are differentiable in a neighborhood $V$ of $M$, the function $\langle X, Y\rangle$ is differentiable on $V$. It is immediate that this definition is equivalent to the other.

It is usual to delete the index $p$ in the function $\langle\ ,\ \rangle_p$ whenever there is no possibility of confusion. The function $g_{ij}\ (= g_{ji})$ is called *the local representation of the Riemannian metric* (or "the $g_{ij}$ of the metric") *in the coordinate system* $\mathbf{x}: U \subset \mathbf{R}^n \to M$. A differentiable manifold with a given Riemannian metric will be called a *Riemannian manifold*.

After introducing any type of mathematical structure, we must introduce a notion of when two objects are the same.

2.2 DEFINITION. Let $M$ and $N$ be Riemannian manifolds. A diffeomorphism $f: M \to N$ (that is, $f$ is a differentiable bijection with a differentiable inverse) is called an *isometry* if:

(1) $\quad \langle u, v\rangle_p = \langle df_p(u), df_p(v)\rangle_{f(p)}$, for all $p \in M$, $u, v \in T_pM$.



2.3 DEFINITION. Let $M$ and $N$ be Riemannian manifolds. A differentiable mapping $f: M \to N$ is a *local isometry* at $p \in M$ if there is a neighborhood $U \subset M$ of $p$ such that $f: U \to f(U)$ is a diffeomorphism satisfying (1).

It is common to say that a Riemannian manifold $M$ is *locally isometric* to a Riemannian manifold $N$ if for every $p$ in $M$ there exists a neighborhood $U$ of $p$ in $M$ and a local isometry $f: U \to f(U) \subset N$.

What follows are some non-trivial examples of the notion of Riemannian manifold.

2.4 EXAMPLE. *The almost trivial example.* $M = \mathbf{R}^n$ with $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, \ldots, 1, \ldots, 0)$. The metric is given by $\langle e_i, e_j\rangle = \delta_{ij}$. $\mathbf{R}^n$ is called *Euclidean space of dimension* $n$ and the Riemannian geometry of this space is metric Euclidean geometry.

2.5 EXAMPLE. *Immersed manifolds.* Let $f: M^n \to N^{n+k}$ be an immersion, that is, $f$ is differentiable and $df_p: T_pM \to T_{f(p)}N$ is injective for all $p$ in $M$. If $N$ has a Riemannian structure, $f$ induces a Riemannian structure on $M$ by defining $\langle u, v\rangle_p = \langle df_p(u), df_p(v)\rangle_{f(p)}$, $u, v \in T_pM$. Since $df_p$is injective, $\langle\ ,\ \rangle_p$ is positive definite. The other conditions of Definition 2.1 are easily verified. This metric on $M$ is then called the metric *induced* by $f$, and $f$ is an *isometric immersion*.

A particularly important case occurs when we have a differentiable function $h: M^{n+k} \to N^k$ and $q \in N$ is a regular value of $h$ (that is, $dh_p: T_pM \to T_{h(p)}N$ is surjective for all $p \in h^{-1}(q)$). It is known then that $h^{-1}(q) \subset M$ is a submanifold of $M$ of dimension $n$; hence, we can put a Riemannian metric on it induced by the inclusion.

For example, let $h: \mathbf{R}^n \to \mathbf{R}$ be given by $h(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^2 - 1$. Then 0 is a regular value of $h$ and $h^{-1}(0) = \{x \in \mathbf{R}^n : x_1^2 + \ldots + x_n^2 = 1\} = S^{n-1}$ is the *unit sphere*of $\mathbf{R}^n$. The metric induced from $\mathbf{R}^n$ on $S^{n-1}$ is called the *canonical metric* of $S^{n-1}$.

2.6 EXAMPLE. *Lie groups.* A *Lie group* is a group $G$ with a differentiable structure such that the mapping $G \times G \to G$ given by $(x, y) \to xy^{-1}$, $x, y \in G$, is differentiable. It follows then that *translations from the left* $L_x$ and *translations from the right* $R_x$ given by: $L_x: G \to G$, $L_x(y) = xy$; $R_x: G \to G, R_x(y) = yx$ are diffeomorphisms.

We say that a Riemannian metric on $G$ is *left invariant* if $\langle u,v\rangle_y = \langle d(L_x)_y u, d(L_x)_y v\rangle_{L_x(y)}$ for all $x,y \in G$, $u,v \in T_yG$, that is, if $L_x$ is an isometry. Analogously, we can define a *right invariant Riemannian metric*. A Riemannian metric on $G$ which is both right and left invariant is said to be *bi-invariant*.

We say that a differentiable vector field $X$ on a Lie group $G$ is *left invariant* if $dL_xX = X$ for all $x \in G$. The left invariant vector fields are completely determined by their values at a single point of $G$. This allows us to introduce an additional structure on the tangent space to the neutral element $e \in G$ in the following manner. To each vector$X_e \in T_eG$ we associate the left invariant $X$ defined by $X_a = dL_aX_e$, $a \in G$. Let $X, Y$ be left invariant vector fields on $G$. Since for each $x \in G$ and for any differentiable function $f$ on $G$,

$$dL_x[X,Y]f = [X,Y](f \circ L_x) = X(dL_xY)f - Y(dL_xX)f =$$

$$= (XY - YX)f = [X,Y]f,$$

we conclude that the bracket of any two left invariant vector fields is again a left invariant vector field. If $X_e, Y_e \in T_eG$, we put $[X_e, Y_e] = [X,Y]_e$. With this operation, $T_eG$ is called the *Lie algebra* of $G$, denoted by $\mathcal{G}$. From now on, the elements in the Lie algebra $\mathcal{G}$ will be thought of either as vectors in $T_eG$ or as left invariant vector fields on $G$.

To introduce a left invariant metric on $G$, take any arbitrary inner product $\langle\ ,\ \rangle_e$ on $\mathcal{G}$ and define

$$(2) \quad \langle u,v\rangle_x = \langle (dL_{x^{-1}})_x(u), (dL_{x^{-1}})_x(v)\rangle_e, \quad x \in G,\ u,v \in T_xG.$$

Since $L_x$ depends differentiably on $x$, this construction actually produces a Riemannian metric, which is clearly left invariant.

In an analogous manner we can construct a right invariant metric on $G$. If $G$ is compact, we will see in Exercise 7 that $G$ possesses a bi-invariant metric.

If $G$ has a bi-invariant metric, the inner product that the metric determines on $\mathcal{G}$ satisfies the following relation: For any $U, V, X \in \mathcal{G}$,

$$(3) \qquad \langle [U,X], V\rangle = -\langle U, [V,X]\rangle.$$



Before proving the relation above, we need some preliminary facts about Lie groups.

For any $a \in G$, let $R_{a^{-1}}L_a \colon G \to G$ be the inner automorphism of $G$ determined by $a$. Such a mapping is a diffeomorphism that keeps $e$ fixed. Thus, the differential $d(R_{a^{-1}}L_a) = Ad(a) \colon \mathcal{G} \to \mathcal{G}$ is a linear map (in fact, it is a homomorphism of the Lie algebra, but we do not need this fact). Explicitly,

$$Ad(a)Y = dR_{a^{-1}}dL_aY = dR_{a^{-1}}Y, \quad \text{for all } Y \in \mathcal{G}.$$

Let $x_t$ be the flow of $X \in \mathcal{G}$. Then, from Proposition 5.4 of Chapter 0,

$$[Y,X] = \lim_{t\to 0} \frac{1}{t}(dx_t(Y) - Y).$$

On the other hand, since $X$ is left invariant, $L_y \circ x_t = x_t \circ L_y$, giving

$$x_t(y) = x_t(L_y(e)) = L_y(x_t(e)) = yx_t(e) = R_{x_t(e)}(y).$$

Therefore, $dx_t = dR_{x_t(e)}$, and

$$[Y,X] = \lim_{t\to 0}\frac{1}{t}(dR_{x_t(e)}(Y) - Y) = \lim_{t\to 0}\frac{1}{t}(Ad(x_t^{-1}(e))Y - Y).$$

Let us now return to the proof of (3). Let $\langle\, ,\, \rangle$ be a bi-invariant metric on a Lie group $G$. Then for any $X, U, V \in \mathcal{G}$,

$$\langle U,V\rangle = \langle dR_{x_t(e)} \circ dL_{x_t^{-1}(e)}U,\ dR_{x_t(e)} \circ dL_{x_t^{-1}(e)}V\rangle =$$

$$= \langle dR_{x_t(e)}U, dR_{x_t(e)}V\rangle.$$

Differentiating the expression above with respect to $t$, recalling that $\langle\, ,\, \rangle$ is bilinear, and setting $t = 0$ in the expression obtained, we conclude that

$$0 = \langle [U,X], V\rangle + \langle U, [V,X]\rangle,$$

which is the equation (3).

The important point about the relation above is that it characterizes the bi-invariant metrics of $G$, in the following sense. If a positive bilinear form $\langle\, ,\, \rangle_e$ defined on $\mathcal{G}$ satisfies the relation (3), then the Riemannian metric defined on $G$ by (2) is bi-invariant. It is not difficult to prove this fact but we will not go into the proof here.

2.7 EXAMPLE. *The product metric.* Let $M_1$ and $M_2$ be Riemannian manifolds and consider the cartesian product $M_1 \times M_2$ with the product structure. Let $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ be the natural projections. Introduce on $M_1 \times M_2$ a Riemannian metric as follows:

$$\langle u, v\rangle_{(p,q)} = \langle d\pi_1 \cdot u, d\pi_1 \cdot v\rangle_p + \langle d\pi_2 \cdot u, d\pi_2 \cdot v\rangle_q,$$

$$\text{for all } (p,q) \in M_1 \times M_2,\ u, v \in T_{(p,q)}(M_1 \times M_2).$$

It is easy to verify that this is really a Riemannian metric on the product. For example, the torus $S^1 \times \cdots \times S^1 = T^n$ has a Riemannian structure obtained by choosing the induced Riemannian metric from $\mathbf{R}^2$ on the circle $S^1 \subset \mathbf{R}^2$ and then taking the product metric. The torus $T^n$ with this metric is called the *flat torus*.

We are now going to show how a Riemannian metric can be used to calculate the lengths of curves.

2.8 DEFINITION. A differentiable mapping $c : I \to M$ of an open interval $I \subset \mathbf{R}$ into a differentiable manifold $M$ is called a (parametrized) *curve*.

Observe that a parametrized curve can admit self-intersections as well as "corners" (Fig. 1).

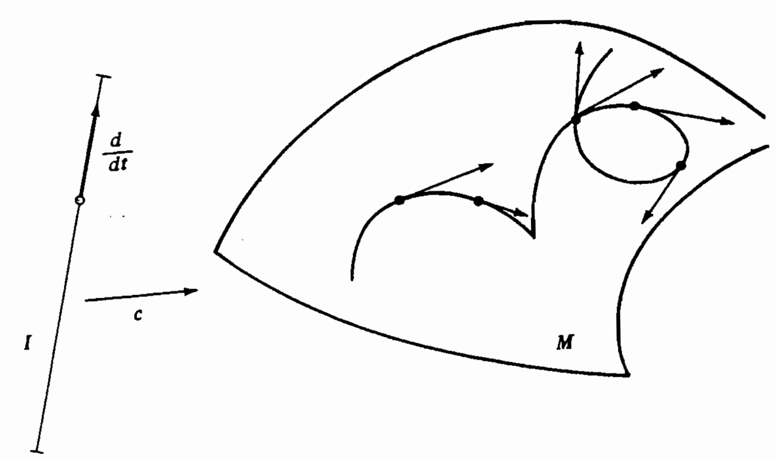


**Figure 1**



2.9 DEFINITION. *A vector field $V$ along a curve $c : I \to M$* is a differentiable mapping that associates to every $t \in I$ a tangent vector $V(t) \in T_{c(t)}M$. To say that $V$ is *differentiable* means that for any differentiable function $f$ on $M$, the function $t \to V(t)f$ is a differentiable function on $I$.

The vector field $dc(\frac{d}{dt})$, denoted by $\frac{dc}{dt}$, is called the *velocity field* (or tangent vector field) of $c$. Observe that a vector field along $c$ cannot necessarily be extended to a vector field on an open set of $M$.

The restriction of a curve $c$ to a closed interval $[a, b] \subset I$ is called a *segment*. If $M$ is a Riemannian manifold, we define the length of a segment by

$$\ell_a^b(c) = \int_a^b \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{1/2} dt.$$

Let us now prove a theorem on the existence of Riemannian metrics.

2.10 PROPOSITION. *A differentiable manifold $M$ (Hausdorff with countable basis) has a Riemannian metric.*

*Proof.* Let $\{f_\alpha\}$ be a differentiable partition of unity on $M$ subordinate to a covering $\{V_\alpha\}$ of $M$ by coordinate neighborhoods. This means (See Chap. 0, Sec. 5) that $\{V_\alpha\}$ is a locally finite covering (i.e., any point of $M$ has a neighborhood $U$ such that $U \cap V_\alpha \neq \phi$ at most for a finite number of indices) and $\{f_\alpha\}$ is a family of differentiable functions on $M$ satisfying:

1) $f_\alpha \geq 0$, $f_\alpha = 0$ on the complement of the closed set $\overline{V}_\alpha$.
2) $\sum_\alpha f_\alpha(p) = 1$ for all $p$ on $M$.

It is clear that we can define a Riemannian metric $\langle\, ,\rangle^\alpha$ on each $V_\alpha$: the metric induced by the system of local coordinates. Let us then set $\langle u, v\rangle_p = \sum_\alpha f_\alpha(p)\langle u, v\rangle_p^\alpha$ for all $p \in M$, $u, v \in T_pM$. It is easy to verify that this construction defines a Riemannian metric on $M$. □

To conclude this chapter, we are going to show how a Riemannian metric permits us to define a notion of volume on a given oriented manifold $M^n$.

As usual we need some preliminary facts. Let $p \in M$ and let $\mathbf{x} : U \subset \mathbf{R}^n \to M$ be a parametrization about $p$ which belongs to a family of parametrizations consistent with the orientation of $M$ (we say that such parametrizations are positive). Consider a positive orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_pM$ and write $X_i(p) = \frac{\partial}{\partial x_i}(p)$ in the basis $\{e_i\}$: $X_i(p) = \sum_{ij} a_{ij}e_j$. Then

$$g_{ik}(p) = \langle X_i, X_k \rangle (p) = \sum_{j\ell} a_{ij}a_{k\ell} \langle e_j, e_\ell \rangle = \sum_j a_{ij}a_{kj}.$$

Since the volume $\mathrm{vol}(X_1(p), \ldots, X_n(p))$ of the parallelepiped formed by the vectors $X_1(p), \ldots, X_n(p)$ in $T_pM$ is equal to $\mathrm{vol}(e_1, \ldots, e_n) = 1$ multiplied by the determinant of the matrix $(a_{ij})$, we obtain

$$\mathrm{vol}(X_1(p), \ldots, X_n(p)) = \det(a_{ij}) = \sqrt{\det(g_{ij})}(p).$$

If $\mathbf{y}: V \subset \mathbf{R}^n \to M$ is another positive parametrization about $p$, with $Y_i(p) = \frac{\partial}{\partial y_i}(p)$ and $h_{ij}(p) = \langle Y_i, Y_j \rangle (p)$,we obtain

$$(4) \qquad \sqrt{\det(g_{ij})}(p) = \mathrm{vol}(X_1(p), \ldots, X_n(p))$$
$$= J\,\mathrm{vol}(Y_1(p), \ldots, Y_n(p)) = J\sqrt{\det(h_{ij})}(p),$$

where $J = \det(\frac{\partial x_i}{\partial y_j}) = \det(d\mathbf{y}^{-1} \circ d\mathbf{x})(p) > 0$ is the determinant of the derivative of the change of coordinates.

Now let $R \subset M$ be a region (an open connected subset), whose closure is compact. We suppose that $R$ is contained in a coordinate neighborhood $\mathbf{x}(U)$ with a positive parametrization $\mathbf{x}: U \to M$, and that the boundary of $\mathbf{x}^{-1}(R) \subset U$ has measure zero in $\mathbf{R}^n$ (observe that the notion of measure zero in $\mathbf{R}^n$ is invariant by diffeomorphism). Let us define the *volume* $\mathrm{vol}(R)$ of $R$ by the integral in $\mathbf{R}^n$

$$(5) \qquad \mathrm{vol}(R) = \int_{\mathbf{x}^{-1}(R)} \sqrt{\det(g_{ij})}\, dx_1 \ldots dx_n.$$

The expression above is well-defined. Indeed, if $R$ is contained in another coordinate neighborhood $y(V)$ with a positive parametrization $\mathbf{y}: V \subset \mathbf{R}^n \to M$, we obtain from the change of



variable theorem for multiple integrals, (using the same notation as in (4),

$$\int_{\mathbf{x}^{-1}(R)} \sqrt{\det(g_{ij})} dx_1 \ldots dx_n$$
$$= \int_{\mathbf{y}^{-1}(R)} \sqrt{\det h_{ij}} dy_1 \ldots dy_n = \mathrm{vol}(R),$$

which proves that the definition given by (5) does not depend on the choice of the coordinate system (here the hypothesis of the orientability of $M$ enters by guaranteeing that $\mathrm{vol}(R)$ does not change sign).

2.11 REMARK. The reader familiar with differential forms will note that equation (4) implies that the integrand in the formula for the volume in expression (5) is a positive differential form of degree $n$, which is usually called a *volume form* (*or volume element*) $\nu$ on $M$. In order to define the volume of a compact region $R$, which is not contained in a coordinate neighborhood it is necessary to consider a partition of unity $\{\varphi_i\}$ subordinate to a (finite) covering of $R$ consisting of coordinate neighborhoods $\mathbf{x}(U_i)$ and to take

$$\mathrm{vol}(R) = \sum_i \int_{x_i^{-1}(R)} \varphi_i \nu.$$

It follows immediately that the expression above does not depend on the choice of the partition of unity.

2.12 REMARK. It is clear that the existence of a globally defined positive differential form of degree $n$ (volume element) leads to a notion of volume on a differentiable manifold. A Riemannian metric is only one of the ways through which a volume element can be obtained.

## EXERCISES

1. Prove that the antipodal mapping $A: S^n \to S^n$ given by $A(p) = -p$ is an isometry of $S^n$. Use this fact to introduce

a Riemannian metric on the real projective space $P^n(\mathbf{R})$ such that the natural projection $\pi: S^n \to P^n(\mathbf{R})$ is a local isometry.

2. Introduce a Riemannian metric on the torus $T^n$ in such a way that the natural natural projection $\pi: \mathbf{R}^n \to T^n$ given by

$$\pi(x_1, \dots, x_n) = (e^{ix_1}, \dots, e^{ix_n}), \quad (x_1, \dots, x_n) \in \mathbf{R}^n,$$

is a local isometry. Show that with this metric $T^n$ is isometric to the flat torus.

3. Obtain an isometric immersion of the flat torus $T^n$ into $\mathbf{R}^{2n}$.

4. A function $g: \mathbf{R} \to \mathbf{R}$ given by $g(t) = yt + x$, $t, x, y \in \mathbf{R}$, $y > 0$, is called a *proper affine function*. The subset of all such functions with respect to the usual composition law forms a Lie group $G$. As a differentiable manifold $G$ is simply the upper half-plane $\{(x, y) \in \mathbf{R}^2; y > 0\}$ with the differentiable structure induced from$\mathbf{R}^2$. Prove that:
   (a) The left-invariant Riemannian metric of $G$ which at the neutral element $e = (0, 1)$ coincides with the Euclidean metric ($g_{11} = g_{22} = 1$, $g_{12} = 0$) is given by $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$, (this is the metric of the non-euclidean geometry of Lobatchevski).
   (b) Putting $(x, y) = z = x + iy$, $i = \sqrt{-1}$, the transformation $z \to z' = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbf{R}$, $ad - bc = 1$ is an isometry of $G$.

   *Hint*: Observe that the first fundamental form can be written as:

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = -\frac{4dzd\bar{z}}{(z - \bar{z})^2}.$$

5. Prove that the isometries of $S^n \subset \mathbf{R}^{n+1}$, with the induced metric, are the restrictions to $S^n$ of the linear orthogonal maps of $\mathbf{R}^{n+1}$.

6. Show that the relation "$M$ is locally isometric to $N$" is not a symmetric relation.

†7. Let $G$ be a compact connected Lie group ($\dim G = n$). The object of this exercise is to prove that $G$ has a bi-invariant Riemannian metric. To do this, take the following approach:



(a) Let $\omega$ be a differential $n$-form on $G$ invariant on the left, that is, $L_x^*\omega = \omega$, for all $x \in G$. Prove that $\omega$ is right invariant.

*Hint*: For any $a \in G$, $\mathbf{R}_a^*\omega$ is left invariant. It follows that $\mathbf{R}_a^*\omega = f(a)\omega$. Verify that$f(ab) = f(a)f(b)$, that is, $f: G \to R - \{0\}$ is a (continuous) homomorphism of $G$ into the multiplicative group of real numbers. Since $f(G)$ is a compact connected subgroup, the conclusion $f(G) = 1$ holds. Therefore $\mathbf{R}_a^*\omega = \omega$.

(b) Show that there exists a left invariant differential $n$-form $\omega$ on $G$.

(c) Let $\langle\, , \rangle$ be a left invariant metric on $G$. Let $\omega$ be a positive differential $n$-form on $G$ which is invariant on the left, and define a new Riemannian metric $\langle\langle\, , \rangle\rangle$ on $G$ by

$$\langle\langle u, v \rangle\rangle_y = \int_G \langle (dR_x)_y u, (dR_x)_y v \rangle_{yx}\, \omega,$$

$$x, y \in G, \quad u, v \in T_y(G).$$

Prove that this new Riemannian metric $\langle\langle\, , \rangle\rangle$ is bi-invariant.