Two Hours

UNIVERSITY OF MANCHESTER

Algebraic Structures MT2262

Tuesday 31st May, 2005
2.00p.m. – 4.00p.m.

Electronic calculators may be used, provided that they cannot store text.

Answer **ALL** five questions in **SECTION A** (40 marks in all)
and
TWO of the three questions in **SECTION B** (20 marks each)
The total number of marks on the paper is 80. A further 20 marks are available from coursework during the semester making a total of 100.

SECTION A

Answer **ALL** five questions

A1. Which of the following binary operations on $\mathbb{R}^\times$ give rise to a group $(\mathbb{R}^\times, o)$? Give reasons for your answers.

(i) $a_o b = -3ab$ (for all $a, b \in \mathbb{R}^\times$),

(ii) $a_o b = |3ab|$ (for all $a, b \in \mathbb{R}^\times$).

[8 marks]

A2. Solve each of the following equations for x in a group G with $a, b, c \in G$.

(i) $a^2 x b^{-3} = c^2 b^4$,

(ii) $a^3 b^2 x^2 c = c^2 a x^3 c$.

[8 marks]

A3. Define what is meant by an *abelian* group. Let $GL_2(\mathbb{R})$ be the group of non-singular 2×2 real matrices under multiplication. Decide whether or not each of the following subgroups of $GL_2(\mathbb{R})$ is abelian or not, giving reasons for your answers.

(i) $H = \left\{ \left(\begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) \middle| \; a \in \mathbb{R} \right\}$,

(ii) $K = \left\{ \left(\begin{array}{cc} b & c \\ 0 & d \end{array} \right) \middle| \; b, c, d \in \mathbb{R}, b \neq 0, d \neq 0 \right\}$.

[8 marks]

A4. Which of the following subsets of $\mathbb{Q}$ are subrings of $\mathbb{Q}$? Give reasons for your answers.

(i) $S_1 = \left\{ \frac{u}{2^m} \middle| \; u \in \mathbb{Z}, m \in \mathbb{Z} \right\}$,

(ii) $S_2 = \left\{ \frac{v^2}{2^n} \middle| \; v \in \mathbb{Z}, n \in \mathbb{Z} \right\}$.

[8 marks]

A5. Find

(i) the number of zero-divisors in $\mathbb{Z}_{100}$

and (ii) the number of units in $\mathbb{Z}_{100}$,

giving reasons for your answers.

[8 marks]

SECTION B

Answer **TWO** of the three questions

B6. Define what is meant by the *order* of an element in a group. Show that if x is an element of finite order in a group and the order of x is the same as the order of x^2 then the order of x is odd.

Find the order of each of the following elements in the given groups, giving reasons for your answers.

(i) $\overline{16} \in \mathbb{Z}_{20}$,

(ii) $\left(\dfrac{\sqrt{3}-i}{2}\right) \in \mathbb{C}^{\times}$,

(iii) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in GL_3(\mathbb{R})$.

[20 marks]

B7. Define what it means for a group to be *cyclic*. Prove that if G is a cyclic group and H is a subgroup of G then H is cyclic.

For each of the following subsets of $\mathbb{Z}_{99}$ (with addition),

(a) find the number of elements in the subset and (b) decide, giving reasons, if the subset is a subgroup of $\mathbb{Z}_{99}$.

(i) $T_1 = \left\{\overline{x} \in \mathbb{Z}_{99} \mid 12\overline{x} = \overline{0}\right\}$,

(ii) $T_2 = \left\{\overline{12y} \;\middle|\; y \in \mathbb{Z}_{99}\right\}$,

(iii) $T_3 = \left\{\overline{k} \;\middle|\; 0 \leq k \leq 98, k \text{ divisible by } 4\right\}$.

[20 marks]

B8. Define what is meant by

(a) a *unit*,

(b) a *zero-divisor* in a commutative ring with identity.

Show that an element in a communtative ring with identity (where $1 \neq 0$) cannot be both a unit and a zero-divisor.

Find an element in $\mathbb{Z} \oplus \mathbb{Z}$ which is neither a unit nor a zero-divisor.

Show that $5 + 2\sqrt{6}$ is a unit in $\mathbb{Z}[\sqrt{6}] = \{m + n\sqrt{6} \mid m, n \in \mathbb{Z}\}$.

Find a unit $r + s\sqrt{6}$ $(r, s \in \mathbb{Z})$ in $\mathbb{Z}[\sqrt{6}]$ with $r > 100$ and $s > 0$.

[20 marks]