or
$$\frac{dF}{d\theta} = \sqrt{\frac{2}{3}}\ \sqrt{(C_2 + 3C_1F - 6F^2 - F^3)}$$

Integrating,
$$\theta = \sqrt{\frac{3}{2}} \int_0^F \frac{dF}{\sqrt{(C_2 + 3C_1F - 6F^2 - F^3)}}, \quad ...(14)$$

which is a solution expressed in terms of an elliptic integral.

Since the order of the differential equation (10) is three, it follows that three boundary conditions are needed to be satisfied by (10). These are :

For convergent channel :

$$\left.\begin{aligned} q_r(\pi+\alpha) = q_r(\pi-\alpha) = 0 \\ \left(\frac{\partial q_r}{\partial \theta}\right)_{(r,\pi)} = 0 \end{aligned}\right\} \quad ...(15)$$

For divergent channel :

$$\left.\begin{aligned} q_r(\alpha) = q_r(-\alpha) = 0 \\ \left(\frac{\partial q_r}{\partial \theta}\right)_{(r,0)} = 0 \end{aligned}\right\} \quad ...(16)$$

Using the above boundary conditions, the constants C_1 and C_2 can be determined and hence the desired solution can be obtained.

14.14. Small Reynold's Number Flows.

Since Navter-Stokes equations are non-linear, their solution in general case in not simple. The main difficulty arises due to presence of non-linear convective terms. These non-linear terms are unimportant when we consider situation with very small Reynolds number. The Reynolds number Ul/ν can be small by reason of the typical velocity U being small or the typical length l being small, or by the kinematic viscosity ν being large. When U is small we have **slow motion** or **creeping motion;** when l is small we have the motion of minute objects, for example **Brownian motion.** Theoretically in creeping flow the Reynolds number is taken to be much less than one. However, it has been seen that the solutions obtained by this process hold good even when Reynolds number is merely less than one.

14.15. Flow Past a Sphere. Stokes Flow. [Himachal Pradesh 2000, 01]

Let a solid sphere of radius a be held fixed in a uniform stream U flowing steadily in the positive direction of the axis of x. Let the fluid be viscous incompressible. Let the flow be steady and axi-symmetric at small Reynolds number. As a first approximation, Stokes neglected the convective terms in the Navier-Stokes equations because they are quadratic in the velocity. Now, the pressure forces must be balanced by viscous forces alone. Hence the equations of motion reduce to

$$0 = -\nabla p + \mu \nabla^2 \mathbf{q} \quad ...(1)$$

and
$$\nabla \cdot \mathbf{q} = 0 \quad ...(2)$$

with boundary conditions :

$$\mathbf{q} = 0 \text{ at } r = a\,;\ \mathbf{q} = (U, 0, 0) \text{ at } r = \infty. \quad ...(3)$$

Taking the divergence of (1) and using (2), we get

$$\nabla^2 p = 0, \quad ...(4)$$

showing that the pressure satisfies the Laplace equation and so the pressure is a harmonic function for small Reynolds number flows.

In spherical polar coordinates (r, θ, ϕ), we choose the axis $\theta = 0$ to lie in the direction of the free stream U. Then the equation of continuity (2) is satisfied if the velocity components are given in terms of stream function ψ by

$$q_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial r}, \quad q_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} \quad ...(5)$$

Using (5), (1) reduces to

$$E^4\psi = 0, \quad ...(6)$$

or
$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0. \quad ...(7)$$

The boundary conditions at the surface of the sphere take the new form :

$$\psi_\theta(1, \theta) = 0, \quad \psi_r(1, \theta) = 0. \quad ...(8)$$

Since the flow is uniform upstream, we have

$$\psi(r, \theta) \sim \tfrac{1}{2} r^2 \sin^2\theta \text{ as } r \to \infty \quad ...(9)$$

and this suggests the trial solution

$$\psi = f(r)\sin^2\theta. \quad ...(10)$$

Substituting this in (7) gives successively

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left[\left(\frac{d^2 f(r)}{dr^2} - \frac{2f(r)}{r^2}\right)\sin^2\theta\right] = 0$$

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)f(r) = 0, \quad ...(11)$$

which is linear homogeneous differential equation of the fourth order. To satisfy (11) by a sum of terms of the form Ar^n, we find

$$[(n-2)(n-3)-2]\,[n(n-1)-2] = 0,$$

so that $n = -1, 1, 2, 4,$ and hence

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4. \qquad ...(12)$$

Condition (9) shows that we must take D = 0. Again conditions (8) show that $A = \frac{1}{4}, B = -\frac{3}{4}$ and $C = \frac{1}{2}$.

∴ From (10) and (12), we have

$$\psi = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta \qquad ...(13)$$

so that

$$q_r = U\left(1 - \frac{3}{2r} + \frac{1}{2r^2}\right)\cos\theta \qquad ...(14)$$

and

$$q_\theta = -U\left(1 - \frac{3}{4r} - \frac{1}{4r^3}\right)\sin\theta. \qquad ...(15)$$

The solution (13) was obtained by Stokes. The first term is the uniform stream and the third term is a dipole at the centre of the sphere, both representing the irrotational flows. The second term, which contains all the vorticity, is known as **Stokeslet.** For non-viscous fluid flow, the Stokeslet is not present and the coefficient of dipole is – 1/2 in place of 1/4. The solution satisfies the surface boundary conditions of the problem. On the other hand it fails to satisfy the boundary condition at infinity. It follows that this expansion breaks down for large r and this breakdown is known as *Whitehead's paradox.*

Let F be the resultant force (drag force) exacted by the fluid on the surface of the sphere in the z-direction. Then

$$F = 2\pi a^2 \int_0^\pi F_{rz} \sin\theta \, d\theta, \qquad ...(16)$$

where F_{rz} is the force per unit area of the spherical surface in the z-direction and is given by

$$F_{rz} = \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta$$

$$= -p\cos\theta + 2\mu\cos\theta\,\frac{\partial q_r}{\partial r} - \mu\sin\theta\left(\frac{\partial q_\theta}{\partial r} + \frac{1}{r}\frac{\partial q_r}{\partial \theta} - \frac{q_\theta}{r}\right)$$

Substituting this in (16) and integrating, we get

$$F = 6\pi\mu U a. \qquad ...(17)$$

This result was first obtained by Stokes and is known as Stokes formula for drag on a sphere.

14.16. Flow Past a Circular Cylinder.

We propose to solve the Stokes equations for uniform flow past a circular cylinder of radius a. For steady flow, the Stokes equations reduce to

$$0 = -\nabla p + \mu\nabla^2 q \qquad ...(1)$$

and

$$\nabla \cdot \mathbf{q} = 0. \qquad ...(2)$$

Taking the curl of (1), we obtain

$$\nabla^2 \mathbf{\Omega} = 0, \qquad ...(3)$$

where

$$\mathbf{\Omega} = \nabla \times \mathbf{q} = \text{vorticity vector} \qquad ...(4)$$

Since in two dimensions the only non-zero component of Ω is ζ (which is the vorticity in the z-direction).

$$\therefore \quad \nabla^2\zeta = 0 \qquad ...(5)$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right), \qquad ...(6)$$

where ψ is the stream function satisfying the continuity equation (2).

Now, the vorticity must satisfy the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \psi = 0. \qquad ...(7)$$

Transforming (7) into cylindrical polar coordinates, we get

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \psi = 0 \qquad ...(8)$$

Since the flow is uniform upstream, we take

$$\psi(r, \theta) \sim r\sin\theta \text{ as } r \to \infty \qquad ...(9)$$

this suggests the trial solution

$$\psi(r, \theta) = f(r)\sin\theta. \qquad ...(10)$$

Substituting this in (8) gives

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)^2 f(r) = 0,$$