

FIGURE 4.12

Ramp response of a mass-spring system

4.4 RESPONSE TO ARBITRARY EXCITATIONS. **THE CONVOLUTION INTEGRAL**

In Ch. 3, we discussed the response of linear time-invariant systems to harmonic and periodic excitations. Then, earlier in this chapter, we considered the response to a unit impulse, a unit step function, a unit ramp function and linear combinations of the latter two. In one form or another, all these excitations have one thing in common, namely, they can all be described as explicit functions of time. The question remains as to how to obtain the response to arbitrary excitations.

For complicated excitations, the general approach is to express them as linear combinations of simpler excitations, sufficiently simple that the response is readily available, or can be produced without much difficulty. In this regard, we should point out that the harmonic response, impulse response, step response and ramp response fall in this category. We used this approach in Sec. 3.9, in which we expressed periodic excitations as Fourier series of harmonic components and then the response to periodic excitations as linear combinations of harmonic responses. Then, in Sec. 4.2 we represented a trapezoidal pulse as a linear combination of step and ramp functions and the response to the trapezoidal pulse as a corresponding linear combination of step and ramp responses. It turns out that the same approach can also be used in the case of arbitrary excitations. There are two ways of deriving the response to arbitrary excitations, depending on the manner in which the excitation function is described. One way is to regard the arbitrary excitation as periodic and represent it by a Fourier series. Then, using a limiting process whereby the period is allowed to approach infinity, so that in essence the function ceases to be periodic and becomes arbitrary, the Fourier series becomes a Fourier integral. This is the frequency-domain representation of functions, which is more suitable for random excitations than for deterministic excitations. This approach is discussed in detail in Ch. 12. The second approach consists of regarding the arbitrary excitation as a superposition of impulses of varying magnitude and applied at different times. This is the time-domain representation of functions, and is the one used in this section.

We consider an arbitrary excitation $F(t)$, such as that depicted in Fig. 4.13, and focus our attention on the contribution to the response of an impulse corresponding to the time interval $\tau < t < \tau + \Delta\tau$. Assuming that the time increment $\Delta\tau$ is sufficiently small

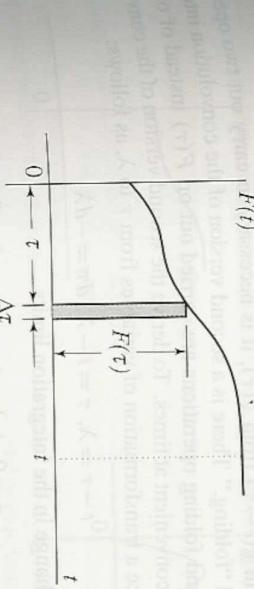


FIGURE 4.13

Arbitrary excitation

that $F(t)$ does not change very much over this time increment, the shaded area in Fig. 4.13 can be regarded as an impulse acting over $\tau < t < \tau + \Delta\tau$ and having the magnitude $F(\tau)\Delta\tau$. Hence, recalling Eq. (4.2), the excitation corresponding to the shaded area can be treated as an impulsive force having the form

$$\hat{F}(\tau)\delta(t - \tau) = F(\tau)\Delta\tau\delta(t - \tau) \quad (4.33)$$

But, as shown in Fig. 4.14, the response of a linear time-invariant system to the impulsive force given by Eq. (4.33) is simply

$$\Delta x(t, \tau) = F(\tau)\Delta\tau g(t - \tau) \quad (4.34)$$

where $g(t - \tau)$ is the impulse response delayed by the time interval τ . Then, regarding the excitation $F(t)$ as a superposition of impulsive forces, we can approximate the response by writing

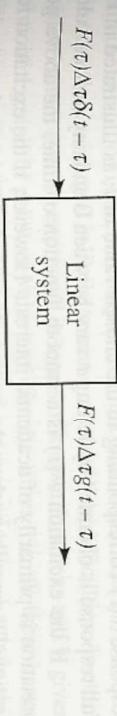
$$x(t) = \sum_{\tau} F(\tau)\Delta\tau g(t - \tau) \quad (4.35)$$

In the limit, as $\Delta\tau \rightarrow 0$, we can replace the summation by integration and obtain the exact response

$$x(t) = \int_0^t F(\tau)g(t - \tau)d\tau \quad (4.36)$$

Equation (4.36) is known as the *convolution integral*, and expresses the response as a superposition of impulse responses. For this reason, Eq. (4.36) is also referred to at times as the *superposition integral*.

We observe that the impulse response in the convolution integral is a function of $t - \tau$, rather than of τ , where τ is the variable of integration. As demonstrated later in

FIGURE 4.14
Block diagram relating the response to an excitation in the form of an impulse of magnitude $F(\tau)\Delta\tau$

this section, to obtain $g(t - \tau)$ from $g(\tau)$, it is necessary to carry out two operations, namely, shifting and “folding.” There is a second version of the convolution integral in which the shifting and folding operations are carried out on $F(\tau)$ instead of on $g(\tau)$, which may be more convenient at times. To derive the second version of the convolution integral, we introduce a transformation of variables from τ to λ , as follows:

$$t - \tau = \lambda, \quad \tau = t - \lambda, \quad d\tau = -d\lambda \quad (4.37)$$

which requires the change in the integration limits

$$\tau = 0 \rightarrow \lambda = t, \quad \tau = t \rightarrow \lambda = 0 \quad (4.38)$$

Introducing Eqs. (4.37) and (4.38) in Eq. (4.36), we obtain

$$x(t) = \int_t^0 F(t - \lambda)g(\lambda)(-d\lambda) = \int_0^t F(t - \lambda)g(\lambda)d\lambda \quad (4.39)$$

which is the second form of the convolution integral. Recognizing that τ in Eq. (4.36) and λ in Eq. (4.39) are mere dummy variables of integration, we can combine Eqs. (4.36) and (4.39) into

$$x(t) = \int_0^t F(\tau)g(t - \tau)d\tau = \int_0^t F(t - \tau)g(\tau)d\tau \quad (4.40)$$

from which we conclude that the convolution integral is symmetric in the excitation $F(t)$ and the impulse response $g(t)$, in the sense that the result is the same regardless of which of the two functions is shifted and folded. The question can be raised as to which form of the convolution integral to use. The choice depends on the nature of the functions $F(t)$ and $g(t)$, and must be the one making the integration task simpler.

The convolution integral lends itself to a geometric interpretation that is not only interesting but at times also quite useful. This interpretation involves the various steps implicit in the evaluation of the integral. To review these steps, we consider the first version of the convolution integral, Eq. (4.36). Figure 4.15a shows an arbitrary excitation $F(\tau)$ and Fig. 4.15b a typical impulse response $g(\tau)$ corresponding to an underdamped mass-damper-spring system, both with t replaced by the variable of integration τ . The first step is to shift the impulse response backward by the time interval t , which yields $g(\tau + t)$, as shown in Fig. 4.15c. The second step is the “folding,” which results in $g(t - \tau)$, as depicted in Fig. 4.15d. The step consists of taking the mirror image of $g(\tau + t) = g(t + \tau)$ with respect to the vertical axis, which amounts to replacing τ by $-\tau$. The third step is to multiply $F(\tau)$ by $g(t - \tau)$, yielding the curve shown in Fig. 4.15e. The final step is the integration of the curve $F(\tau)g(t - \tau)$, which is the same as determining the area under the curve in Fig. 4.15e. The result is one point on the response $x(t)$ corresponding to the chosen value of t , as illustrated in Fig. 4.15f. The full response is obtained by letting t vary between 0 and any desired value.

If the excitation $F(t)$ is a smooth function of time, the above geometric interpretation is primarily of academic interest. However, if the excitation function is only sectionally smooth, such as the rectangular pulse of Fig. 4.8, then the limits of integration in the convolution integral must be chosen judiciously. In this regard, the preceding geometric interpretation is vital to a successful determination of the response, as the

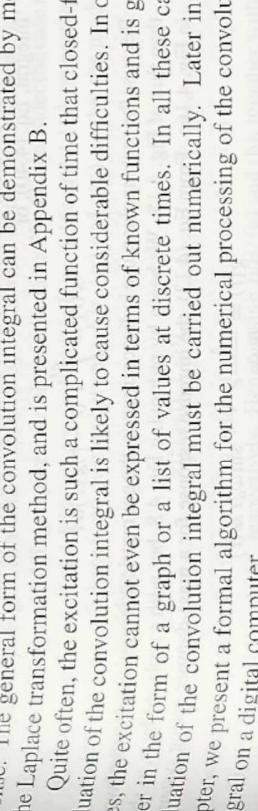
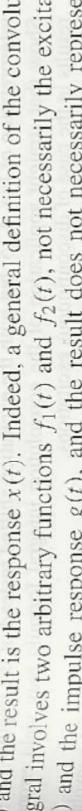
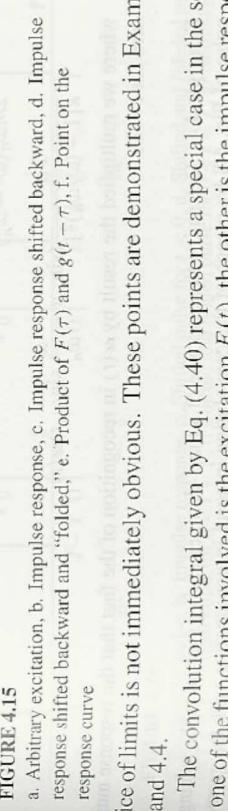
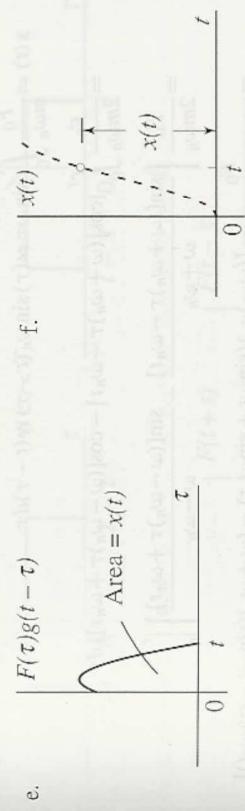
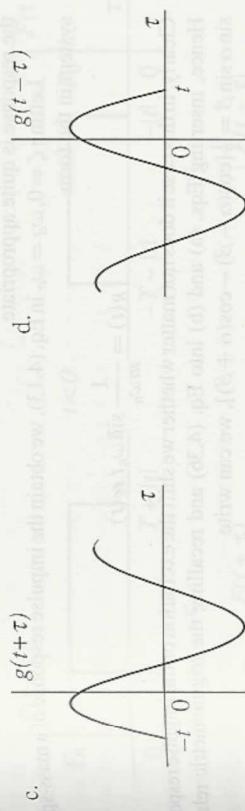
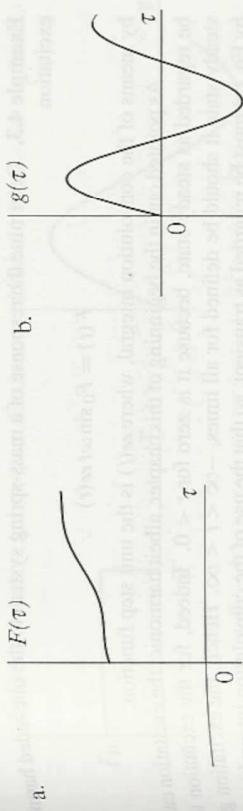


FIGURE 4.15

a. Arbitrary excitation, b. Impulse response, c. Impulse response shifted backward, d. Impulse response shifted backward and “folded,” e. Product of $F(\tau)$ and $g(t - \tau)$, f. Point on the response curve

choice of limits is not immediately obvious. These points are demonstrated in Examples 4.3 and 4.4.

The convolution integral given by Eq. (4.40) represents a special case in the sense that one of the functions involved is the excitation $F(t)$, the other is the impulse response $g(t)$ and the result is the response $x(t)$. Indeed, a general definition of the convolution integral involves two arbitrary functions $f_1(t)$ and $f_2(t)$, not necessarily the excitation $F(t)$ and the impulse response $g(t)$, and the result does not necessarily represent a response. The general form of the convolution integral can be demonstrated by means of the Laplace transformation method, and is presented in Appendix B.

Quite often, the excitation is such a complicated function of time that closed-form evaluation of the convolution integral is likely to cause considerable difficulties. In other cases, the excitation cannot even be expressed in terms of known functions and is given either in the form of a graph or a list of values at discrete times. In all these cases, evaluation of the convolution integral must be carried out numerically. Later in this chapter, we present a formal algorithm for the numerical processing of the convolution integral on a digital computer.

Example 4.3. Determine the response of a mass-spring system to the one-sided harmonic excitation

$$F(t) = F_0 \sin \omega_n t \varepsilon(t)$$

by means of the convolution integral, where $\varepsilon(t)$ is the unit step function.

As pointed out in the beginning of this chapter, albeit harmonic, the excitation can be regarded as steady state, because it is zero for $t < 0$. Indeed, for the excitation to be steady state it should be defined for all times, $-\infty < t < \infty$. Hence, the excitation given by Eq. (a) must be regarded as transient, so that the use of the convolution integral to obtain the response is quite appropriate.

Letting $\zeta = 0$, $\omega_d = \omega_n$ in Eq. (4.13), we obtain the impulse response of a mass-spring system in the form

$$g(t) = \frac{1}{m\omega_n} \sin \omega_n t \varepsilon(t)$$

Clearly, in this case it does not matter whether we shift the excitation or the impulse response. Hence, inserting Eqs. (a) and (b) into Eq. (4.36) and recalling the trigonometric relation $\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$, we can write

$$\begin{aligned} x(t) &= \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n \tau \varepsilon(\tau) \sin \omega_n(t - \tau) \varepsilon(t - \tau) d\tau \\ &= \frac{F_0}{2m\omega_n} \int_0^t [\cos((\omega + \omega_n)\tau - \omega_n t) - \cos((\omega - \omega_n)\tau + \omega_n t)] d\tau \end{aligned}$$

$$\begin{aligned} &= \frac{F_0}{2m\omega_n} \left\{ \frac{\sin[(\omega + \omega_n)\tau - \omega_n t]}{\omega + \omega_n} - \frac{\sin[(\omega - \omega_n)\tau + \omega_n t]}{\omega - \omega_n} \right\} \Big|_0^t \\ &= \frac{F_0}{2m\omega_n(\omega^2 - \omega_n^2)} [(\omega - \omega_n)(\sin \omega t + \sin \omega_n t) - (\omega + \omega_n)(\sin \omega t - \sin \omega_n t)] \\ &= \frac{F_0}{k[1 - (\omega/\omega_n)^2]} (\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t) \varepsilon(t) \end{aligned} \quad (c)$$

where we multiplied the result by $\varepsilon(t)$ in recognition of the fact that the response must be zero for $t < 0$.

Example 4.4. Determine the response of a mass-damper-spring system to the rectangular pulse of Fig. 4.8 by means of the convolution integral in conjunction with the geometric interpretation of Fig. 4.15, but with the excitation shifted instead of the impulse response. Show that straight application of the convolution integral formula may yield erroneous results in the case of discontinuous excitations.

The problem statement calls for the use of the second version of the convolution integral, Eq. (4.40), or

$$x(t) = \int_0^t F(t - \tau) g(\tau) d\tau \quad (a)$$

Because the rectangular pulse is discontinuous, the geometric interpretation is more involved than that of Fig. 4.15. We begin by redrawing Fig. 4.8, but with t replaced by τ , as shown in Fig. 4.16a. From Eq. (4.13), the impulse response for a mass-damper-spring system is

$$g(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_n t \varepsilon(t)$$

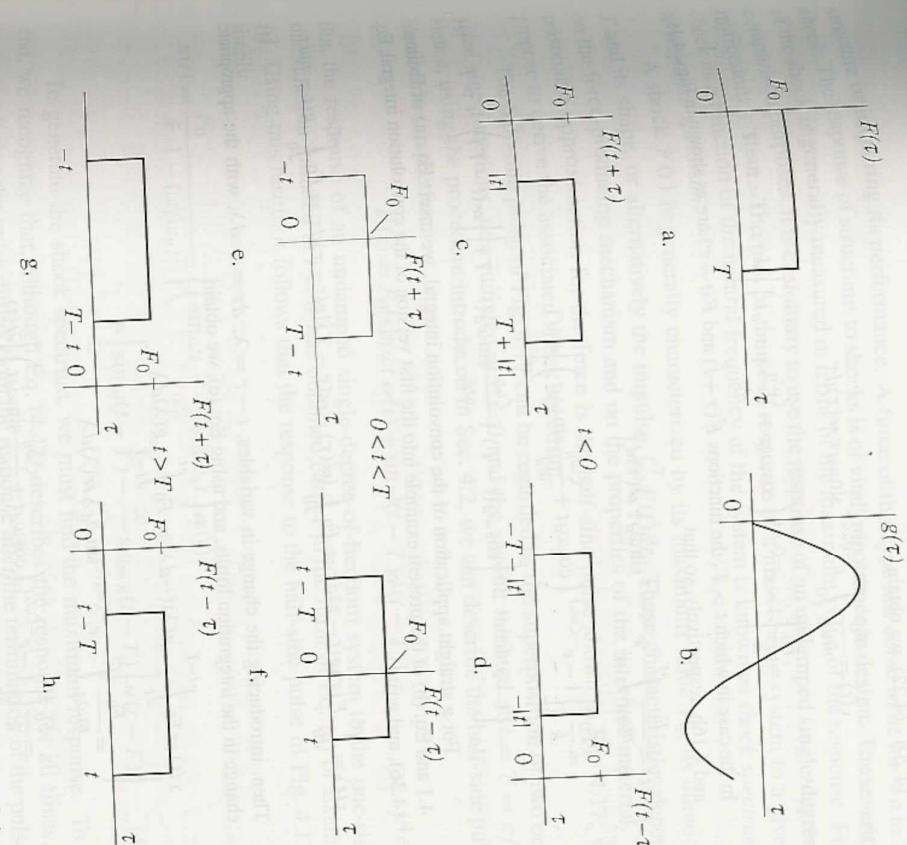


FIGURE 4.16

a. Rectangular pulse. b. Impulse response. c. Shifted pulse for $t < 0$. d. Shifted and folded pulse for $t > T$. e. Shifting and folding require careful consideration. We first consider the case $t < 0$, for $t < 0$, f. Shifted and folded pulse for $0 < t < T$, g. Shifted pulse for $t > T$

for $t > T$, h. Shifted and folded pulse for $t > T$

It is displayed in Fig. 4.16b with t replaced by τ . In view of the discontinuous nature of $F(\tau)$, the shifting and folding require careful consideration. We first consider the case $t < 0$, in which case $F(\tau + t)$ and $F(\tau - t)$ are as shown in Figs. 4.16c and 4.16d, respectively. It is clear that for $t < 0$ there is no overlap between $F(\tau + t)$ and $F(\tau - t)$, so that the product of the two is zero, yielding

$$x(t) = 0, t < 0 \quad (c)$$

which is to be expected. For $0 < t < T$, the functions $F(\tau + t)$ and $F(\tau - t)$ are as depicted in Figs. 4.16e and Fig. 4.16f, respectively. Hence, using Eqs. (a) and (b), together with Eqs.