Although methods of solving linear and quadratic equations were known since the Babylonians, mathematicians struggled for centuries to find a formula that solved cubic equations to find the zeros of cubic functions. In 1545, a method of solving a cubic equation of the form $x^3 + mx = n$, developed by Niccolo Tartaglia, was published in the *Ars Magna,* a work by Girolamo Cardano. The formula for finding the one real solution of the equation is

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{\frac{-n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

(*Source:* Gullberg, J., *Mathematics from the Birth of Numbers,* W. W. Norton & Company, 1997.)

For Discussion or Writing

Use the formula to solve the equation $x^3 + 9x = 26$ for the one real solution.

ercises

rue **3.** false; −2 is
ltiplicity 4. **4.** true
es **7.** no **8.** no
yes **11.** no
yes **14.** no
. yes
$x - 2)(2x - 5) \cdot$
. $f(x) = (x - 2) \cdot$
- 1) **19.** $f(x) =$
- 1) (2x − 1)
$x + 5)(6x - 1) \cdot$
. $f(x) = (x + 4) \cdot$
+ 1) **22.** $f(x) =$
- 1) (2x + 3)
$x - 3i)(x + 4) \cdot$
. $f(x) = (x + i) \cdot$
+ 1) **25.** $f(x) =$
] (2x − 1) (x + 3)
$x - (-2 + i)] \cdot$
+ 1) **27.** $f(x) =$
1) (x − 3)
$x + 1)^3(2x - 5)$
30. $\frac{-5 \pm \sqrt{5}}{2}$
32. $\frac{-2 \pm \sqrt{2}}{2}$
34. $-i, -5 \pm i$

Concept Check Decide whether each statement is true *or* false. *If false, tell why.*

1. Since $x - 1$ is a factor of $f(x) = x^6 - x^4 + 2x^2 - 2$, we can conclude that $f(1) = 0$.

2. Since $f(1) = 0$ for $f(x) = x^6 - x^4 + 2x^2 - 2$, we can conclude that $x - 1$ is a factor of $f(x)$.

3. For $f(x) = (x + 2)^4(x - 3)$, 2 is a zero of multiplicity 4.

4. Since $2 + 3i$ is a zero of $f(x) = x^2 - 4x + 13$, we can conclude that $2 - 3i$ is also a zero.

Use the factor theorem and synthetic division to decide whether the second polynomial is a factor of the first. See Example 1.

5. $x^3 - 5x^2 + 3x + 1;\ x - 1$

6. $x^3 + 6x^2 - 2x - 7;\ x + 1$

7. $2x^4 + 5x^3 - 8x^2 + 3x + 13;\ x + 1$

8. $-3x^4 + x^3 - 5x^2 + 2x + 4;\ x - 1$

9. $-x^3 + 3x - 2;\ x + 2$

10. $-2x^3 + x^2 - 63;\ x + 3$

11. $4x^2 + 2x + 54;\ x - 4$

12. $5x^2 - 14x + 10;\ x + 2$

13. $x^3 + 2x^2 - 3;\ x - 1$

14. $2x^3 + x + 2;\ x + 1$

15. $2x^4 + 5x^3 - 2x^2 + 5x + 6;\ x + 3$

16. $5x^4 + 16x^3 - 15x^2 + 8x + 16;\ x + 4$

Factor $f(x)$ into linear factors given that k is a zero of $f(x)$. See Example 2.

17. $f(x) = 2x^3 - 3x^2 - 17x + 30;\ k = 2$

18. $f(x) = 2x^3 - 3x^2 - 5x + 6;\ k = 1$

19. $f(x) = 6x^3 + 13x^2 - 14x + 3;\ k = -3$

20. $f(x) = 6x^3 + 17x^2 - 63x + 10;\ k = -5$

21. $f(x) = 6x^3 + 25x^2 + 3x - 4;\ k = -4$

22. $f(x) = 8x^3 + 50x^2 + 47x - 15;\ k = -5$

35. **(a)** $\pm 1, \pm 2, \pm 5, \pm 10$ **(b)** $-1, -2, 5$ **(c)** $f(x) = (x+1)(x+2)(x-5)$
36. **(a)** $\pm 1, \pm 2, \pm 4, \pm 8$ **(b)** $-4, -2, 1$ **(c)** $f(x) = (x+4)(x+2) \cdot (x-1)$ **37.** **(a)** $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30$ **(b)** $-5, -3, 2$ **(c)** $f(x) = (x+5)(x+3)(x-2)$
38. **(a)** $\pm 1, \pm 2, \pm 4, \pm 8$ **(b)** $-2, -1, 4$ **(c)** $f(x) = (x+2)(x+1) \cdot (x-4)$ **39.** **(a)** $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{1}{6}$ **(b)** $-4, -\frac{1}{3}, \frac{3}{2}$
(c) $f(x) = (x+4)(3x+1) \cdot (2x-3)$ **40.** **(a)** $\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{4}{5}, \pm \frac{8}{5}, \pm \frac{1}{15}, \pm \frac{2}{15}, \pm \frac{4}{15}, \pm \frac{8}{15}$ **(b)** $-4, -\frac{2}{5}, \frac{1}{3}$
(c) $f(x) = (x+4)(5x+2) \cdot (3x-1)$ **41.** **(a)** $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{1}{6}, \pm \frac{1}{8}, \pm \frac{3}{8}, \pm \frac{1}{12}, \pm \frac{1}{24}$
(b) $-\frac{3}{2}, -\frac{2}{3}, \frac{1}{2}$
(c) $f(x) = 2(2x+3)(3x+2) \cdot (2x-1)$ **42.** **(a)** $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \pm \frac{1}{6}, \pm \frac{1}{8}, \pm \frac{1}{12}, \pm \frac{1}{24}, \pm \frac{2}{3}, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm \frac{3}{8}, \pm \frac{4}{3}, \pm \frac{8}{3}$
(b) $-\frac{3}{2}, -\frac{4}{3}, -\frac{1}{2}$ **(c)** $f(x) = 2(2x+3)(3x+4)(2x+1)$
43. $0, \pm \frac{\sqrt{7}}{7} i$ **44.** -1 (multiplicity 2), 1 (multiplicity 3), $\pm\sqrt{10}$ **45.** $2, -3, 1, -1$
46. 0 (multiplicity 2), $-1+\sqrt{2}$, $-\frac{5}{2}$ **47.** -2 (multiplicity 5), 1 (multiplicity 5), $1-\sqrt{3}$ (multiplicity 2)

23. $f(x) = x^3 + (7-3i)x^2 + (12-21i)x - 36i;\ k = 3i$
24. $f(x) = 2x^3 + (3+2i)x^2 + (1+3i)x + i;\ k = -i$
25. $f(x) = 2x^3 + (3-2i)x^2 + (-8-5i)x + (3+3i);\ k = 1+i$
26. $f(x) = 6x^3 + (19-6i)x^2 + (16-7i)x + (4-2i);\ k = -2+i$
27. $f(x) = x^4 + 2x^3 - 7x^2 - 20x - 12;\ k = -2$ (multiplicity 2)
28. $f(x) = 2x^4 + x^3 - 9x^2 - 13x - 5;\ k = -1$ (multiplicity 3)

For each polynomial function, one zero is given. Find all others. See Examples 2 and 6.

29. $f(x) = x^3 - x^2 - 4x - 6;\ 3$
30. $f(x) = x^3 + 4x^2 - 5;\ 1$
31. $f(x) = x^3 - 7x^2 + 17x - 15;\ 2 - i$
32. $f(x) = 4x^3 + 6x^2 - 2x - 1;\ \frac{1}{2}$
33. $f(x) = x^4 + 5x^2 + 4;\ -i$
34. $f(x) = x^4 + 10x^3 + 27x^2 + 10x + 26;\ i$

For each polynomial function, ***(a)*** *list all possible rational zeros,* ***(b)*** *find all rational zeros, and* ***(c)*** *factor f(x). See Example 3.*

35. $f(x) = x^3 - 2x^2 - 13x - 10$
36. $f(x) = x^3 + 5x^2 + 2x - 8$
37. $f(x) = x^3 + 6x^2 - x - 30$
38. $f(x) = x^3 - x^2 - 10x - 8$
39. $f(x) = 6x^3 + 17x^2 - 31x - 12$
40. $f(x) = 15x^3 + 61x^2 + 2x - 8$
41. $f(x) = 24x^3 + 40x^2 - 2x - 12$
42. $f(x) = 24x^3 + 80x^2 + 82x + 24$

For each polynomial function, find all zeros and their multiplicities.

43. $f(x) = 7x^3 + x$
44. $f(x) = (x+1)^2(x-1)^3(x^2-10)$
45. $f(x) = 3(x-2)(x+3)(x^2-1)$
46. $f(x) = 5x^2(x+1-\sqrt{2})(2x+5)$
47. $f(x) = (x^2+x-2)^5(x-1+\sqrt{3})^2$
48. $f(x) = (7x-2)^3(x^2+9)^2$

Find a polynomial function of degree 3 with real coefficients that satisfies the given conditions. See Example 4.

49. Zeros of -3, 1, and 4; $f(2) = 30$
50. Zeros of 1, -1, and 0; $f(2) = 3$
51. Zeros of -2, 1, and 0; $f(-1) = -1$
52. Zeros of 2, -3, and 5; $f(3) = 6$
53. Zero of -3 having multiplicity 3; $f(3) = 36$
54. Zero of 4 having multiplicity 2 and zero of 2 having multiplicity 1; $f(1) = -18$

Find a polynomial function of least degree having only real coefficients with zeros as given. See Examples 4–6.

55. $5 + i$ and $5 - i$
56. $7 - 2i$ and $7 + 2i$
57. 2 and $1 + i$
58. $-3, 2, -i$, and $2 + i$
59. $1 + \sqrt{2}, 1 - \sqrt{2}$, and 1
60. $1 - \sqrt{3}, 1 + \sqrt{3}$, and 1
61. $2 + i, 2 - i, 3$, and -1
62. $3 + 2i, -1$, and 2
63. 2 and $3 + i$
64. -1 and $4 - 2i$
65. $1 - \sqrt{2}, 1 + \sqrt{2}$, and $1 - i$
66. $2 + \sqrt{3}, 2 - \sqrt{3}$, and $2 + 3i$
67. $2 - i$ and $6 - 3i$
68. $5 + i$ and $4 - i$
69. $4, 1 - 2i$, and $3 + 4i$
70. $-1, 1 + \sqrt{2}, 1 - \sqrt{2}$, and $1 + 4i$
71. $1 + 2i$ and 2 (multiplicity 2)
72. $2 + i$ and -3 (multiplicity 2)