Three Hours

UNIVERSITY OF MANCHESTER

Multiple Integrals, Vector Field Theory and Tensors MT2121

Tuesday 15th January, 2002
9.45a.m. – 12.45p.m.

The use of electronic calculators is **NOT** permitted.

Answer **ALL** seven questions in **SECTION A** (39 marks in all)
and
THREE of the four questions in **SECTION B** (12 marks each)
The total number of marks on this paper is 75. A further 25 marks are available from the coursework during the semester, making a total of 100.
This module is worth 20 credits, so marks gained will be counted twice.

You may use the following formulae in Sections A and B without proof:-

In cylindrical polar coordinates (r, θ, z), where $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}} + F_z\hat{\mathbf{z}}$,

$$(1) \qquad \operatorname{div}\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

and

$$(2) \qquad \operatorname{curl}\mathbf{F} = \left[\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right]\hat{\mathbf{r}} + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]\hat{\mathbf{z}}.$$

In spherical polar coordinates (R, θ, ϕ), where $\mathbf{F} = F_R\hat{\mathbf{R}} + F_\theta\hat{\boldsymbol{\theta}} + F_\phi\hat{\boldsymbol{\phi}}$,

$$(3) \qquad (R\sin\theta)\operatorname{div}\mathbf{F} = \frac{\sin\theta}{R}\frac{\partial}{\partial R}(R^2F_R) + \frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{\partial}{\partial \phi}F_\phi$$

and

$$(4) \qquad (R\sin\theta)\operatorname{curl}\mathbf{F} = \left[\frac{\partial}{\partial \theta}(\sin\theta F_\phi) - \frac{\partial}{\partial \phi}F_\theta\right]\hat{\mathbf{R}}$$

$$+ \left[\frac{\partial}{\partial \phi}F_R - \frac{\partial}{\partial R}(R\sin\theta F_\phi)\right]\hat{\boldsymbol{\theta}} + \left[\frac{\partial}{\partial R}(RF_\theta) - \frac{\partial}{\partial \theta}F_R\right]\sin\theta\,\hat{\boldsymbol{\phi}}.$$

SECTION A

Answer **ALL** seven questions

A1. For each of the following equations, state whether it is valid or not (**in terms of the suffices only**). If an equation is not valid, state the reason and one way to correct it.

(i) $d'_{ij}b'_{jk} = \alpha_{ip}\alpha_{jq}d_{pq}\alpha_{jq}\alpha_{kr}b_{qr}$;

(ii) $C_{lmn} = D_{luv}K_{nvm}X_u$;

(iii) $P_{zy} = Q_{zx}R_{xy} + S_zT_{yu}$.

[5 marks]

A2. Verify the formula

$$\epsilon_{zmn}\epsilon_{zst} = \delta_{ms}\delta_{nt} - \delta_{mt}\delta_{ns}$$

in the case $m = 1$ and $n = 2$. Do this by finding the value of the left-hand and right-hand sides in each of the four cases (i) $s = 3$, (ii) $t = 3$, (iii) $s = t \neq 3$ and (iv) all other possible values of s and t. [5 marks]

A3. With the aid of a clear sketch of the region of integration, reverse the order of integration in the integral

$$I = \int_0^1 \int_0^{\sqrt{x}} \frac{x^2}{1+y^3}\, dy\, dx$$

to show that $I = 1/4$.

[You may use the formula $(1-y^6) = (1-y^3)(1+y^3)$.] [5 marks]

A4. The surfaces S_1 and S_2 are respectively the sphere $x^2 + y^2 + (z-2a)^2 = 2a^2$ and the cone $z^2 = x^2 + y^2$ (where a is a constant). You are given that the sphere sits inside and touches the cone; show that the circle where they touch lies in the plane $z = a$ and has radius a.

Find the volume trapped between S_1 and S_2.

[Your solution should include a clear sketch of the region under consideration.]

[6 marks]

A5. Write down the formula for grad P, where $P = P(R, \theta, \phi)$ is given in spherical polar coordinates, and use the given formula (3) for the divergence to write down $\nabla^2 P$.

You are given that P has the form

$$P(R, \theta, \phi) = f(R) + \ln(\sin\theta)$$

for some function $f(R)$; show that

$$\nabla^2 P = \frac{1}{R^2}\left\{\frac{d}{dR}\left(R^2 \frac{df}{dR}\right) - 1\right\}.$$

You are given also that $\mathbf{F} =$ grad P is solenoidal, and that $f(R) = f'(R) = 0$ on $R = 1$. Find the function $f(R)$. [6 marks]

A6. The vector field $\mathbf{F}$ is given by

$$\mathbf{F} = (y^2 + 2\lambda y - x^2)\mathbf{i} + 2x(y+1)\mathbf{j} + 7x\mathbf{k}$$

(where λ is a constant). The loop C in the (x, y) plane consists of the following three parts: (i) the parabola $y = x^2$ from the origin to the point $(1, 1, 0)$, (ii) the straight line $y = 1$ to the point $(0, 1, 0)$ and (iii) the straight line $x = 0$ back to the origin.

By direct calculation of the contributions from the three parts of C, evaluate the loop integral $\oint_c \mathbf{F} \cdot d\mathbf{r}$.

From your results, say whether you can deduce that $\mathbf{F}$ is, or is not, a conservative field in the two cases $\lambda = \pm 1$. [7 marks]

A7. Write down the Cartesian components of curl $\mathbf{A}$, and use these to show that

$$\text{div}\,(\text{curl}\,\mathbf{A}) \equiv 0$$

for any vector field $\mathbf{A}$ with sufficiently smooth components.

Assume that, for some suitably smooth functions $\phi(x, y)$ and $\psi(x, y)$, $\mathbf{A} = \psi(x, y)\mathbf{k}$ and curl $\mathbf{A} =$ grad ϕ. Write down the Cartesian components of the last equation, and show that each of ϕ and ψ satisfies Laplace's equation, that is $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$.

[5 marks]

SECTION B

Answer **THREE** of the four questions

B8. (a) You are given that the rotation matrix A, from the unprimed frame to another frame, has components α_{pq} which satisfy $\alpha_{pq}\alpha_{pr} = \delta_{qr} = \alpha_{qp}\alpha_{rp}$.

You are also given that B and C are both second-order tensors and have components b_{ij} and c_{ij} in the unprimed frame.

Define in component form the tensor product $D = BC$ and show that D is a fourth-order tensor. Show also that the quantity E, which has components $e_{ik} = b_{ij}c_{jk}$ in the unprimed frame, is another second-order tensor. [6 marks]

(b) The primed frame is now taken to be the one obtained from the unprimed frame by a $\frac{\pi}{2}$ rotation about the $0x_1$ axis (so that $0x'_3$ coincides with $0x_2$). Sketch the two sets of axes, and find all the components α_{ij} of the rotation matrix A; here α_{ij} is the cosine of the angle between $0x'_i$ and $0x_j$.

Show that A is orthogonal, state the transformation law applicable to B, and use this to find the four components $b'_{12}, b'_{13}, b'_{22}$ and b'_{23} of B in the primed frame. [6 marks]

B9. (a) Show that the Jacobian of the transformation

$$u = \frac{x^2}{y} \text{ and } v = \frac{y^2}{x}$$

is a constant. Use this transformation to show that

$$\int\int_R (3y^3 - 2x^3) dx\, dy = \frac{14}{9},$$

where R is the region bounded between the parabolas $y = \sqrt{x}$ and $y = \frac{1}{3}x^2$ and the straight line $y = x$. [Your solution should include a sketch of the region of integration R, and its image under the transformation.] [6 marks]

(b) The surface S is that part of the spheroid

$$x^2 + y^2 + \frac{1}{4}z^2 = 3a^2$$

which lies inside the paraboloid $az = x^2 + y^2$; here a is a constant. Sketch the surface S and the paraboloid by drawing their intersections with the plane $y = 0$.

Show that

$$\int\int_S z\, dS = 2\sqrt{3} \int\int_R \sqrt{a^2 + x^2 + y^2}\, dx\, dy,$$

where R is a region of the (x, y) plane you should find, and hence evaluate the surface integral (using plane polar coordinates in the plane integral). [6 marks]

B10. (a) State the Divergence Theorem, being careful to explain any notation you use and any conditions that must apply.

The vector field $\mathbf{B}$ is given by

$$\mathbf{B} = R\cos\theta(\cos\theta\,\hat{\mathbf{R}} - \sin\theta\,\hat{\boldsymbol{\theta}})$$

in spherical polar coordinates (R, θ, ϕ). This field exists in a region which includes the hemisphere $x^2 + y^2 + z^2 \leq a^2, z \geq 0$. By direct evaluation of the two surface integrals, find the flux $\int\int \mathbf{B} \cdot \hat{\mathbf{n}}\, dS$ of $\mathbf{B}$ out of the (closed) surface of the hemisphere. [Hint: this surface consists of two parts, a hemispherical cap and a flat base, which are respectively $R =$ const and $\theta =$ const.]

Now use the given formula (3) for the divergence to show that div $\mathbf{B} = 1$, and confirm your result for the flux by an application of the Divergence Theorem. [8 marks]

(b) The vector field $\mathbf{F}$ is given by

$$\mathbf{F} = (\lambda x^2 y^2 + 2xe^{-y})\mathbf{i} + (2x^3 y - x^2 e^{-y} + 1)\mathbf{j}$$

where λ is a constant. For a certain value of λ, which you should find, the line integral

$$S(\mathbf{r}) = \int_{\mathbf{0}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}$$

is independent of the path from $\mathbf{0}$ to $\mathbf{r}$. Find $S(\mathbf{r})$ for this particular value of λ.

[4 marks]

B11. State Stokes' Theorem, being careful to explain any notation you use and paying particular attention to the direction of all vectors.

In spherical polar coordinates (R, θ, ϕ) the vector field $\mathbf{B}$ is given by

$$\mathbf{B} = R^2 \cos^3\theta\,\hat{\mathbf{R}} - R^2 \sin\theta(\cos^2\theta + 1)\,\hat{\boldsymbol{\theta}}.$$

By direct calculation, find the flux $\int\int \mathbf{B} \cdot \hat{\mathbf{n}}\, dS$ of $\mathbf{B}$ through the part of the sphere $x^2 + y^2 + z^2 = a^2$ with $z \geq h$ (where $a > h > 0$). [The substitution $c = \cos\theta$ may be useful.]

Now use the given formula (4) to show that $\mathbf{B} =$ curl $\mathbf{A}$, where

$$\mathbf{A} = \frac{1}{4} R^3 \sin\theta(\cos^2\theta + 1)\,\hat{\boldsymbol{\phi}},$$

and use Stokes' Theorem to confirm your result for the flux. [12 marks]