2. Legendre polynomials, so instrumental in the development of the Gaussian quadrature, can be defined in an alternative way via the Rodrigues formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \qquad (n = 0, 1, 2, \ldots). \tag{1}$$

One easily observes that P_n is a polynomial of degree n

(a) Assume that the function f and its first n derivatives are continuous on the interval $[-1, 1]$. Using integration by parts and the definition (1), prove the following formula

$$\int_{-1}^{1} f(x) P_n(x)\, dx = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} (x^2 - 1)^n \frac{d^n f(x)}{dx^n}\, dx. \tag{2}$$

(b) Using formula (2), show that

$$\int_{-1}^{1} x^m P_n(x)\, dx = 0 \qquad (m = 0, 1, \ldots, n - 1). \tag{3}$$

(c) Again, using formula (2), compute $\int_{-1}^{1} x^n P_n(x)\, dx$.