Bessel Functions of the First Kind of Half-Integer Order

Bessel functions of the first kind, $y(x) = J_\nu(x)$, are solutions to the differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)\, y = 0 \tag{1}$$

that converge at $x = 0$ for $\nu \geq 0$. The constant ν is a called the *order* of the equation and $J_\nu(x)$. The order ν can be any real or complex number, but the most common and important values are integers or half-integers. These functions have many applications to physics and engineering. For more background on Bessel functions and their applications, see [1, 2], for example, and the links and references therein.

Solutions to linear differential equations with nonconstant coefficients like (1) are generally obtained by using a power series method and are therefore typically expressed as power series. In the special case when $\nu = \pm 1/2$, the series solutions can be simplified to yield

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{x}}\,\sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{x}}\,\cos x. \tag{2}$$

Furthermore, it can be shown that the functions $J_\nu(x)$ satisfy the recursion relation

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x). \tag{3}$$

In the problem below, we use (2) and (3) to derive a recursive method for writing $J_\nu(x)$ as a combination of polynomials and trigonometric functions when ν is a half-integer. The main idea of the solution is to use (3) in conjunction with a proof by induction. The point is that (3) shows us how to build $J_{\nu+1}(x)$ from Bessel functions of lower order.

Problem Statement

1. Show that there exist exist polynomials $P_n(z)$ and $Q_n(z)$ for positive integers n such that the Bessel function $J_{n+1/2}(x)$ satisfies

$$J_{n+1/2}(x) = P_n(x^{-1/2}) \sin x + Q_n(x^{-1/2}) \cos x. \tag{4}$$

2. Find $P_1(z)$ and $Q_1(z)$.

References

[1] Boyce, William E. and DiPrima, Richard C. *Elementary Differential Equations and Boundary Value Problems.* 7-th edition. John Wiley & Sons, Inc. 2001.

[2] `http://en.wikipedia.org/wiki/Bessel_function.`