18.3 Separation of Variables

18.3.1. The method of separation of variables. We explain the method of separation of variables by a sequence of examples.

EXAMPLE 1. Consider the diffusion problem

$$L[u] = \alpha^2 u_{xx} - u_t = 0, \qquad (0 < x < L,\ 0 < t < \infty) \tag{1a}$$

$$u(0,t) = u_1, \quad u(L,t) = u_2, \qquad (0 < t < \infty) \tag{1b}$$

$$u(x,0) = f(x), \qquad (0 < x < L) \tag{1c}$$

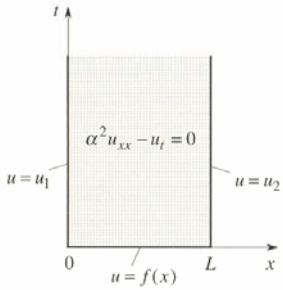

Figure 1. The problem (1).

that is derived in Section 18.2 and that governs the temperature field $u(x,t)$ in a rod with insulated lateral surface (or in a wall or slab of thickness L); see Fig. 1.

According to the method of **separation of variables** we begin by seeking solutions of (1a) in the product form

$$\boxed{u(x,t) = X(x)T(t).} \tag{2}$$

Putting (2) into (1a) gives

$$\alpha^2 X''T = XT', \tag{3}$$

where primes denote ordinary differentiation. To separate the variables, divide both sides of (3) by XT and obtain

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}. \tag{4}$$

Actually, we divided by α^2 as well, but whether we have $1/\alpha^2$ on the right-hand side of (4) or α^2 on the left-hand side will not affect the final result. Observe that (4) is of the form

$$F(x) = G(t). \tag{5}$$