Since x and t are independent variables, $G(t)$ does not vary with x. But $F(x) = G(t)$, so $F(x)$ does not vary with x either. Hence $F(x)$ is a constant. From (5), $G(t)$ is a constant also, the same constant.* Thus,

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = \text{constant} = -\kappa^2, \tag{6}$$

say, where we have written κ^2 for convenience because we will soon need to take the square root of that quantity. Motivation for including the minus sign in (6) is given in the end-of-example comments.

The beauty of the separation procedure is that in place of the *partial* differential equation $\alpha^2 u_{xx} = u_t$ we now have two *ordinary* differential equations,

$$\frac{X''}{X} = -\kappa^2 \qquad \text{and} \qquad \frac{1}{\alpha^2}\frac{T'}{T} = -\kappa^2,$$

or,

$$X'' + \kappa^2 X = 0, \tag{7a}$$

$$T' + \kappa^2\alpha^2 T = 0. \tag{7b}$$

The *separation constant* κ remains to be determined. Solving (7a,b) gives

$$X = A\cos\kappa x + B\sin\kappa x, \tag{8a}$$

$$T = Ce^{-\kappa^2\alpha^2 t}. \tag{8b}$$

However, observe that (8a) is the general solution of (7a) only in the event that $\kappa \neq 0$, for if $\kappa = 0$ then the $\sin\kappa x$ term drops out. Since we do not yet know the value(s) of κ, we must allow for the possibility that the value $\kappa = 0$ will be needed. Setting $\kappa = 0$ in (7a) gives $X'' = 0$, with the general solution $X = D + Ex$, so we replace (8a) by the two-tier statement

$$X = \begin{cases} A\cos\kappa x + B\sin\kappa x, & \kappa \neq 0 \\ D + Ex, & \kappa = 0. \end{cases} \tag{9}$$

Apparently, we don't need to revise (8b) the way we revised (8a) because (8b) is a general solution of (7b) whether κ is zero or not. However, having already comitted ourselves in (9) to the separate treatment of these two cases, we replace (8b) by the two-tier statement †

$$T = \begin{cases} Fe^{-\kappa^2\alpha^2 t}, & \kappa \neq 0 \\ G, & \kappa = 0. \end{cases} \tag{10}$$

Thus far we have discussed the product solutions

$$u = XT = (D + Ex)G \tag{11}$$

corresponding to $\kappa = 0$, and

$$u = XT = (A \cos \kappa x + B \sin \kappa x)Fe^{-\kappa^2 \alpha^2 t} \tag{12}$$

for any $\kappa \neq 0$. Since D, E, G, A, B, F are arbitrary constants, we can combine DG as H and EG as I and simplify (11) as

$$u = H + Ix$$

for $\kappa = 0$, and we can combine AF as J and BF as K and simplify (12) as

$$u = (J \cos \kappa x + K \sin \kappa x)e^{-\kappa^2 \alpha^2 t}$$

for $\kappa \neq 0$. Since (1a) is linear, the sum of these solutions must also be a solution, so we can write

$$u = H + Ix + (J \cos \kappa x + K \sin \kappa x)e^{-\kappa^2 \alpha^2 t}, \tag{13}$$