(a) Consider the unit circle $x^2 + y^2 = 1$, and line with slope -1 and equation $x + y = d$.

(i) Show that when $d = \pm\sqrt{2}$, the line is tangent to the circle.

(ii) For $d \in (-\sqrt{2}, \sqrt{2})$, the line meets the circle in two points. What are they?

(iii) Consider the triangle T whose vertices are the two points found in (ii) and the origin. Find the values of d for which the area of T is largest.

(b) The hyperbolic function $\operatorname{cotanh} x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{2e^{-x}}{e^x - e^{-x}} + 1 = \dfrac{2e^x}{e^x - e^{-x}} - 1$ has domain $\mathbb{R}$, range $\mathbb{R} - \{0\}$ and properties:

$$\operatorname{cotanh}' x = -\operatorname{cosech}^2 x \qquad \text{and} \qquad \operatorname{cotanh}^2 x + 1 = \operatorname{cosech}^2 x$$

(i) Explain, using the properties of the exponential functions, or of $\sinh x$ and $\cosh x$ or their reciprocals $\operatorname{cosech} x$ and $\sec x$, why $\operatorname{cotanh} x$ is an odd function, which is decreasing on $\mathbb{R} - \{0\}$, and approaches $\pm\infty$ as $x \to 0^+$ or $x \to 0^-$ respectively.

(ii) Determine whether the limits $\lim\limits_{x \to -\infty} \operatorname{cotanh} x$ and $\lim\limits_{x \to -\infty} \operatorname{cotanh} x$ exist.

(iii) Use the result of (i) to explain why $\operatorname{cotanh} x$, has an inverse function, and state its domain.

(iv) Use the above properties of $\operatorname{cotanh} x$ to find an expression for the derivative of the inverse function $\operatorname{cotanh}^{-1} x$ obtained in (ii).

(v) Confirm that $\operatorname{cotanh} x$, and its inverse function each have negative derivatives throughout their domains.