2. Show that the risk-neutral probabilities in the Cox-Ross-Rubinstein model are given by equations (6.15).

(See attached pages for relevant information from the textbook)

is preferable to taking a position in the futures contract and getting them only in the future. For example, there may be benefits to owning the commodity in the case of shortages, or to keep a production process running, or because of their consumption value. In such situations, the value of the futures contract becomes smaller than in the case in which there are no benefits in holding the commodity, so we conclude

$$F(t) \leq [S(t) + \bar{U}(t)]e^{r(T-t)}$$

or

$$F(t) \leq S(t)e^{(r+u)(T-t)}$$

as the case may be. As a measure of how much smaller the futures price becomes, we define the **convenience yield** (which represents the value of owning the commodity) as the value y for which

$$F(t)e^{y(T-t)} = [S(t) + \bar{U}(t)]e^{r(T-t)}$$

or

$$F(t)e^{y(T-t)} = S(t)e^{(r+u)(T-t)}$$

In general, any type of cost or benefit associated with a futures contract (whether it is a storage cost, a dividend, or a convenience yield) is called **cost of carry.** We define it as the value c for which

$$F(t) = S(t)e^{(r+c)(T-t)}$$

6.3 Risk-Neutral Pricing

We have hinted before that the absence of arbitrage implies that the contingent claims that can be replicated by a trading strategy could be priced by using expectations under a special, risk-neutral probability measure. In the present section we explain why this is the case. The main results of this section are summarized in figure 6.1.

6.3.1 Martingale Measures; Cox-Ross-Rubinstein (CRR) Model

The modern approach to pricing financial contracts, as well as to solving portfolio-optimization problems, is intimately related to the notion of **martingale probability measures.** As we shall see, prices are expected values, but not under the "real-world" or "true" probability; rather, they are expected values under an "artificial" probability, called **risk-neutral probability** or **equivalent martingale measure (EMM).**

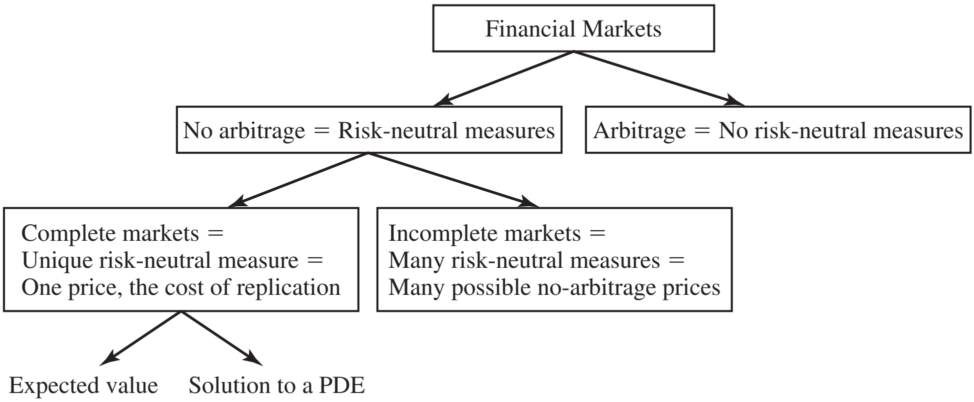

Figure 6.1
Risk-neutral pricing: no arbitrage, completeness, and pricing in financial markets.

We first recall the notion of a martingale: Consider a process X whose values on the interval $[0, s]$ provide information up to time s. Denote by E_s the conditional expectation given that information. We say that a process X is a martingale if

$$E_s[X(t)] = X(s), \qquad s \leq t \tag{6.14}$$

(We implicitly assume that the expected values are well defined.) This equation can be interpreted as saying that the best possible prediction for the future value $X(t)$ of the process X is the present value $X(s)$. Or, in profit/loss terminology, a martingale process, on average, makes neither profits nor losses. In particular, taking unconditional expected values in equation (6.14), we see that $E[X(t)] = E[X(s)]$. In other words, *expected values of a martingale process do not change with time.*

Recall our notation $\bar{A}$ that we use for any value A discounted at the risk-free rate. We say that a probability measure is a **martingale measure** for a financial-market model *if the discounted stock prices $\bar{S}_i$ are martingales.*

Let us see what happens in the Cox-Ross-Rubinstein model with one stock. Recall that in this model the price of the stock at period $t+1$ can take only one of the two values, $S(t)u$ or $S(t)d$, with u and d constants such that $u > 1 + r > d$, where r is the constant risk-free rate, and we usually assume $d < 1$. At every point in time t, the probability that the stock takes the value $S(t)u$ is p, and, therefore, $q := 1 - p$ is the probability that the stock will take the value $S(t)d$. Consider first a single-period setting. A martingale measure will be given by probabilities p^* and $q^* := 1 - p^*$ of up and down moves, such that the discounted stock price is a martingale:

$$S(0) = \bar{S}(0) = E^*[\bar{S}(1)] = p^* \frac{S(0)u}{1+r} + (1-p^*)\frac{S(0)d}{1+r}$$

Here, $E^* = E_0^*$ denotes the (unconditional, time $t = 0$) expectation under the probabilities p^*, $1 - p^*$. Solving for p^* we obtain

$$p^* = \frac{(1+r)-d}{u-d}, \qquad q^* = \frac{u-(1+r)}{u-d} \tag{6.15}$$

We see that the assumption $d < 1+r < u$ guarantees that these numbers are indeed positive probabilities. Moreover, these equations define the only martingale measure with positive probabilities. Furthermore, p^* and q^* are strictly positive, so that events that have zero probability under the "real-world" probability measure also have zero probability under the martingale measure, and vice versa. We say that the two **probability measures are equivalent** and that the probabilities p^*, q^* form an **equivalent martingale measure** or **EMM.**

In order to make a comparison between the actual, real probabilities p, $1-p$ and the risk-neutral probabilities p^*, $1-p^*$, introduce the mean return rate μ of the stock as determined from

$$S(0)(1+\mu) = E[S(1)]$$

Then a calculation similar to the preceding implies that we get expressions analogous to equations (6.15):

$$p = \frac{(1+\mu)-d}{u-d}, \qquad 1-p = \frac{u-(1+\mu)}{u-d} \tag{6.16}$$

Thus we can say that

the risk-neutral world *is the world in which there is no compensation for holding the risky assets, hence in which the expected return rate of the risky assets is equal to the risk-free rate r.*

We want to make a connection between the price of a contingent claim and the possibility of replicating the claim by trading in other securities, as discussed in chapter 3. Denote now by δ the number of shares of stock held in the portfolio and by x the initial level of wealth, $X(0) = x$. The rest of the portfolio, $x - \delta S(0)$, is invested in the bank at the risk-free rate r. Therefore, from the budget constraint of the individual, the discounted wealth $\bar{X}(1)$ at time 1 is given by

$$\bar{X}(1) = \delta\bar{S}(1) + x - \delta S(0)$$