An overview of single-period business decision-making

Managerial decision-making using accounting information

We focus here on a particular type of managerial decision-making: *profit maximization*. Although there are many types of managerial decisions that we will seemingly ignore (e.g., human resource decisions, product quality decisions, etc. ad infinitum), it can usually be shown that such decisions can easily be characterized as profit-maximizing decisions. Rather than discuss the wide variety of profit maximization decisions generally made by managers, we will focus only the simplest types of decisions in order to obtain the basic intuition underlying quantitative managerial decision-making based—at least in part—on accounting information.

Those who have actually observed managers in the real world are perhaps likely to question whether managers actually maximize profits through their decisions. After all, managers seem to rarely use the term *profit maximization*, or use quantitative methods to make decisions as shown in this course. Managers rather tend to speak in terms that suggest they are simply trying to *improve* their firm's profits, not *maximize* them. Theoretical profit maximization can be reconciled with such practical profit improvement by recognizing that, since the real world is so complex, managers must maximize expected profits through a series of decisions over time as they receive new information. So they are in fact maximizing expected profits through a more-or-less continual series of decisions, each of which also happens to improve expected profits.

We have studied how accounting information can be used in developing estimates of expected costs at different levels of production and sales—i.e., *cost function estimation*—used in planning and controlling costs. It turns out that cost function estimation is more generally applicable in managerial decision-making aimed at maximizing a firm's profits. To see this notice the only difference between a linear cost function, $C(Q) = vQ + f$, and a linear profit function,

$$\begin{aligned}\pi(Q) &= R(Q) - C(Q) \\ &= pQ - (vQ + f) \, , \\ &= (p - v)Q - f\end{aligned}$$

is the addition of the linear revenue function $R(Q) = pQ$ where *p* is the average selling price per product unit. It can be seen from this relationship that cost minimization is a special case of profit maximization. As always, it's important to remember that although we will use *linear* functions in this course, profit and cost functions are usually *non-linear* in the real world. This means that decisions made on the basis of linear profit and cost functions are only appropriate within the *relevant range* of unit production and sales volume; i.e., the range over which linear approximations to non-linear profit and cost functions are reasonably accurate.

Our general study of profit maximization will encompass two somewhat different types of decisions:

(1) *single period decisions*, which commonly affect only one period (e.g., whether or not to make and sell a product using the firm's existing excess production capacity); and

(2) *multiple period decisions*, which commonly affect two or more periods (e.g., whether or not construct a new manufacturing plant and purchase new manufacturing equipment so that the firm can make and sell new products).

The basic difference between the two types of decisions is that single period decisions are associated with investments having expected payoffs only in the same period the investment is made, while multi-period decisions are associated with investments having expected payoffs over two or more future periods. As we will see later in Unit 3, the only rational way to make multi-period decisions is to restate future payoffs (i.e., roughly speaking, profits) to their *present value* at the date of the decision. More specifically we will study the following profit maximization decisions:

	Single-period decisions	**Multi-period decisions**
Cost minimization	*Outsourcing*	*Capital asset investments*
Profit maximization	*Special order acceptance* *Product selection* *Sell-or-process-further*	*All*

The "*All*" entry in the table refers to the idea that any of the other decisions could be formulated as a multi-period profit maximization decision (as long as payoffs are expected over more than one future period).

An algorithm for linear profit function maximization

Before beginning our study of profit maximization it is worthwhile thinking about some of the unique aspects of linear profit functions and what is means, precisely, to say such a function is maximized by some decision. It turns out that linear functions are always maximized (or minimized) by choosing either the maximum or minimum value of a variable. So, for example, if a linear profit function suggests there is some positive marginal profit from selling each unit of a product, then the profit maximizing choice of the number of product units to make and sell is *infinity*—a very large number, that is—at least according to the linear profit function.

Such unrealistic results suggest there must be some sort of *constraints* that make the solutions more reasonable (e.g., production or sales volume cannot exceed a certain number of product units). Recalling that linear profit functions make sense only over some relevant range of production and sales volume, it makes sense that solutions to linear profit maximization problems should consider constraints on production or sales volume. A basic algorithm for solving such *constrained linear profit maximization* problems can be written as:

(1) identify the model appropriately framing the decision (e.g., linear profit function);
(2) identify the variable to be chosen to maximize profits (e.g., product unit sales volume);
(3) obtain or estimate values of parameters included in the model (e.g., product sales price);
(4) identify constraints on the decision variable while considering its relevant range; and
(5) solve for the profit maximizing value of the decision variable subject to the constraints.

The algorithm might seem somewhat abstract, so it's worthwhile considering a simple profit maximization problem as an example:

<u>Example</u>: Applying an algorithm for simple constrained linear profit maximization

A firm's manager has developed a linear profit function for use in decision-making. Profits are a function of units produced and sold, *Q*, and the manager obtained parameter estimates through discussions with the firm's sales staff ($\hat{p}=20$) and estimating the firm's cost function ($\hat{v}=10$ and $\hat{f}=1000$) from monthly production and accounting records with $Q=[0,120]$:

$$\hat{\pi}(Q)=(\underbrace{20}_{\hat{p}}-\underbrace{10}_{\hat{v}})Q-\underbrace{1000}_{\hat{f}} .$$

The manager has also had discussions with production staff indicating that it is not possible to make more than 120 units per period. The manager wants to determine the profit maximizing *Q* using the algorithm presented above (mainly to illustrate how to use the algorithm); so …

(1) *identify the general model appropriately framing the decision*:

Since the manager wants to maximize profits and a linear profit function is presumably appropriate between 0 and 120 units, the general model would be

$$\pi(Q)=(p-v)Q-f .$$

(2) *identify the decision variable to be chosen to maximize profits*:

Since the manager wants to the find the profit maximizing unit sales volume the decision variable is *Q*, which will be chosen at the value Q^* that maximizes expected profit:

$$\max \pi(Q)=(p-v)Q^*-f .$$

(3) *obtain or estimate values of parameters included in the model*:

As discussed, the manager has obtained estimated parameter values that can be substituted into the profit function:

$$\hat{\pi}(Q)=(20-10)Q-1000 .$$

(4) *identify constraints on the decision variable while considering relevant range*:

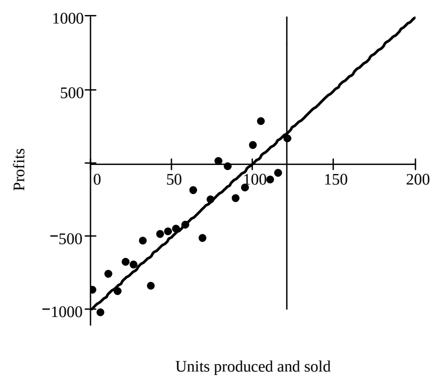
The manager has identified the maximum unit production of 120, so it follows that *Q* must be constrained: $Q\leq 120$. (The firm's linear cost function was estimated using unit production volumes of $Q=[0,120]$, so it is perhaps reasonable to conclude $Q\leq 120$ is within the relevant range for the firm.)

(5) *solve for the profit maximizing value of the decision variable*:

Since the firm's expected *contribution margin per unit* is positive, $\hat{p}-\hat{v}=20-10=10>0$, it follows that profits are maximized at the highest number of product units possible to produce and sell; in this case it is $Q^{*}=120$. To see this more clearly notice that

$$\left.\begin{array}{l} \hat{\pi}(Q^{*}=120)=(20-10)120-1000=1200-1000=200 \\ \hat{\pi}(Q=119)=(20-10)119-1000=1190-1000=190 \\ \quad\vdots \\ \hat{\pi}(Q=0)=(20-10)0-1000=0-1000=-1000 \end{array}\right\} \Rightarrow \hat{\pi}(Q<Q^{*})<\hat{\pi}(Q^{*}),$$

as clearly shown in the following graph where expected profits are lower for all $Q<Q^{*}=120$ (which is indicated by the thin vertical line between 100 and 150 units):



Notice that as long as the expected sales volume (denoted $\overline{Q}$) is above the **break-even point** (denoted Q_{BEP}), which is the point at which profits are zero, then the firm would generally want to make and sell the product since profits are positive. The break-even point can be found for any linear profit function by setting profits equal to zero (break-even) and solving for Q_{BEP}:

$$\underbrace{\hat{\pi}(Q)}_{0}=(\hat{p}-\hat{v})Q-\hat{f} \quad \Rightarrow \quad Q_{BEP}=\hat{f}\,/\,(\hat{p}-\hat{v})=10000\,/\,(20-10)=100\,.$$

Also notice that in the process of studying the application of the algorithm, we also demonstrated a unique characteristic of linear (profit) functions: If the slope of the function is *positive*, it is maximized at the *highest* possible (constrained) value of the variable; if the slope is *negative*, the function is maximized at the *lowest* variable value.

As we have seen, maximizing or minimizing any *single* linear function with constraints is fairly simple (and perhaps even trivial). In more general settings where production and sales volume constraints don't make sense (e.g., where the firm is considering investments in plant and equipment that will change production capacity and the firm's cost and profit functions), profit maximization can become substantially more complex; though the intuition gained from studying simpler decisions still holds.

It is, in effect, common to use linear functions for profit maximization and cost minimization in a way that avoids some of the complexity by formulating decisions as choices between two linear functions. We discuss this in the following sections. Going forward we will generally omit the “hat” notation for estimates, $\hat{\pi}(Q) = (\hat{p} - \hat{v})Q - \hat{f}$, and regard all parameter values (*p*,*v*,*f*) as estimates to simplify notation.

Cost minimization

Cost minimization is a component of profit maximization as can be seen from looking carefully at the expression,

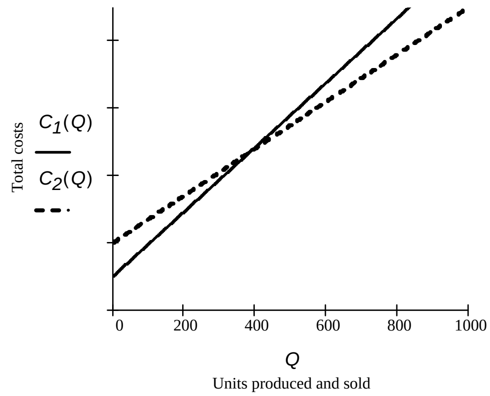
$$\max \pi(Q) = \max R(Q) - \min C(Q),$$

which simply says maximized profit is equal to maximized revenues less minimized costs. That is, maximizing expected profits is the same thing as choosing actions that will result in the highest expected revenues and the lowest expected costs. In this section we will focus exclusively on cost minimization decisions as a prelude to more general types of profit maximization decisions in later sections and chapters.

Cost minimization by choosing a cost function. In the last section we saw minimizing a linear function, such as the cost function $C(Q) = vQ + f$, is very straightforward: If Q is the decision variable, then costs are minimized when Q = 0. Obviously, knowing this is not helpful in managerial decision-making. How then can linear functions be meaningfully used in decision-making? It turns out that decision-making using linear functions basically comes down to *choosing between two or more functions*. That is, if a manager wants to minimize expected costs by choosing some alternative method of making a product (or way of purchasing components for making the product), then the manager is essentially choosing the expected cost function from a set of cost functions,

$$\left\{ \begin{array}{c} C_1(Q) = v_1 Q + f_1 \\ C_2(Q) = v_2 Q + f_2 \\ \vdots \\ C_k(Q) = v_k Q + f_k \end{array} \right\},$$

that results in the lowest expected cost. So the manager is choosing from a set of cost functions representing the alternative methods $1, 2, ..., k$ of making the product. (In this course we will usually focus only on choosing between *two* cost or profit functions.) It is important to keep in mind that the cost function that minimizes expected costs depends on the level of unit production and sales, Q. To see this consider a graph of two, alternative cost functions for making a given product:

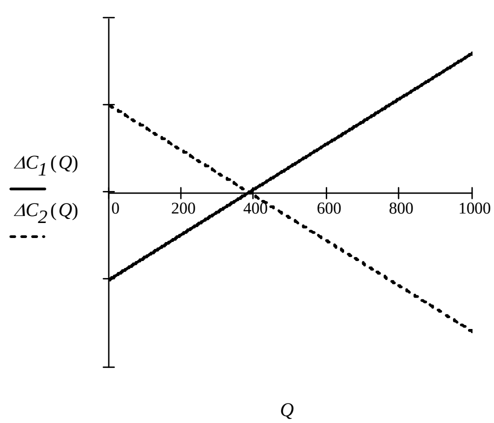


Notice the cost function C_1 has a higher slope that C_2, and C_2 has a higher intercept than C_1. So under the production method represented by C_1, the manager expects variable cost per unit to be higher—and the fixed costs to be lower—than in the method represented by C_2. This means that at production/sales levels below what is called an **indifference point** ($Q_{\text{indifference}}$), in the example shown in the graph C_1 is lower than C_2; above the indifference point C_2 is lower than C_1. It follows that *the cost minimizing choice of an expected cost function depends on Q*.

Cost difference functions. It is sometimes convenient to formulate cost minimization decisions in terms of the difference between two cost functions:

$$\begin{aligned}\Delta C_1(Q) &= C_1(Q) - C_2(Q) \\ &= (v_1 Q + f_1) - (v_2 Q + f_2) \\ &= (v_1 - v_2)Q + (f_1 - f_2) \\ &= \Delta v_1 Q + \Delta f_1\end{aligned}$$

where this *cost difference function* is interpreted as the excess of cost function 1, $C_1(Q)$, over cost function 2, $C_2(Q)$. The precise interpretation is important: The following graph is based on the two cost functions used in the previous graph, where one cost difference function is defined as $\Delta C_1(Q) = C_1(Q) - C_2(Q)$ and the other is defined as $\Delta C_2(Q) = C_2(Q) - C_1(Q)$.



The cost difference function $\Delta C_1(Q)$ has a *positive slope* and crosses the *Q* axis at about 400, and $\Delta C_2(Q)$ has a *negative slope* and crosses the *Q* axis at the same point. The point at which the functions cross the *Q* axis is the indifference point between the two functions. Most importantly the graph says that C_1 minimizes costs when $Q < Q_{\text{indifference}}$, and C_2 minimizes costs when $Q > Q_{\text{indifference}}$.

Avoidable and unavoidable fixed costs. It usually seems strange to most people that fixed costs can change as a result of a manager's decision, as implied above by the Δf_1 in the cost difference function. After all, aren't fixed costs actually fixed? The answer is, Yes, *by definition fixed costs are fixed with respect units produced and sold, but not with respect to managers' decisions*. That is, fixed costs depend on the production methods used by the firm—which can change based on managers' decisions—but once production methods are in place, fixed costs are essentially independent of production and sales volume.

The dimension of production method choice adds to the complexity of cost minimization and requires managers to distinguish between *avoidable fixed costs* and *unavoidable fixed costs*. **Avoidable fixed costs** are simply costs that do not vary with production and sales levels once the decision to use a particular production/sales method is in place. In contrast, **unavoidable fixed costs**—as their name implies—are fixed costs that cannot be avoided, usually because the firm is required to incur a fixed cost as a result of contracts (e.g., rental contracts for manufacturing facilities and equipment). Such unavoidable fixed costs are referred to as **sunk costs** in economics. From a practical perspective, a fixed cost is avoidable if it changes as a result of a decision (other than to make more or less units of a product); otherwise the fixed cost is unavoidable.

Single period cost minimization decisions. Cost minimization decisions that do not affect the firm's cash flows for more than one period can be analyzed using the simple mathematical tools described above. Although it is perhaps unrealistic to think of *any* managerial decision as affecting only one period, we will ignore this temporarily and briefly examine two common cost minimization decisions as if they only affect a single period:

- *outsourcing production activities*, and
- *investments in capital assets*.

Outsourcing production activities refers to the firm entering into a contract where another firm will perform the production activities in exchange for some price; usually, a price per product unit. The decision of whether or not to make investments in capital assets (e.g., production facilities and equipment) has to do with either expanding or renewing a firm's production capacity. Rather than exploring the actual multi-period nature of such decisions in detail, we will mention the limitations of our analyses and study multi-period decisions in a later chapter.

<u>Cost minimization</u>: Outsourcing production activities

Firms periodically study different production methods hoping to find lower cost methods, which in turn can make the firms more competitive; allowing lower prices while maintaining profitability. Firms sometimes find outsourcing production lowers expected production costs when a combination of conditions like geographic differences in labor costs and low-cost transportation exist. Outsourcing can also reduce risk: If the firm is uncertain about the extent to which it can use its costly production facilities, then outsourcing might reduce risk by allowing the firm to sell-off the facilities and avoid (fixed) costs necessary to maintain them. That is, outsourcing might allow the firm to avoid the possibility that uncertainty over expected sales volume will result in negative profits resulting from fixed costs.

<u>Outsourcing decision with avoidable fixed costs.</u> Consider a firm that makes electrical generators used as a component in its automobiles. Based on the firm's historical experience and engineering estimates, expected variable material and labor costs are \$40 per unit, expected variable overhead costs are \$10 per unit, and *avoidable* fixed overhead costs are \$10,000 per period. The firm has an offer from a supplier to make the generators for \$64 per unit and wants to determine whether it should enter into the outsourcing contract or continue to make the generator itself.

Given information about the firm's expected cost function if it continues to make the generators, and about the outsourcing contract, the firm's manager is basically choosing between the two cost functions,

$$C_m(Q) = v_m Q + f_m = 50Q + 10000$$
$$C_o(Q) = v_o Q + f_o = 64Q$$

where the subscript *m* denotes *make* and *o* denotes *outsource*. The cost difference function

$$\begin{aligned}\Delta C_o(Q) &= C_o(Q) - C_m(Q) \\ &= (v_o - v_m)Q + (f_o - f_m) \\ &= (64-50)Q + (0-10000) \\ &= 14Q - 10000\end{aligned}$$

can then be analyzed to assist in making the outsourcing decision. The indifference point, at which the firm would be indifferent between making the generators and outsourcing their production, can be found by setting the cost difference function to zero and solving for *Q*:

$$0 = 14Q - 10000 \quad \Rightarrow \quad Q_{\text{indifference}} = 10000/14 \cong 714\,.$$

Since the cost difference function represents the excess of outsourcing costs over manufacturing costs, and since its slope is positive, it follows that outsourcing would lower expected costs if $Q < Q_{\text{indifference}}$ and would increase expected costs if $Q > Q_{\text{indifference}}$. So this means that the choice of whether or not depends on the expected unit production and sales volume, $\bar{Q}$:

$$\bar{Q} < 714 \Rightarrow \text{outsource production}$$
$$\bar{Q} \geq 714 \Rightarrow \textit{do not} \text{ outsource production}$$

Outsourcing decision with unavoidable fixed costs. It's important to notice that if the fixed costs were not avoidable, then the decision would be quite different:

$$\begin{aligned}\Delta C_o(Q) &= C_o(Q) - C_m(Q)\\ &= (v_o - v_m)Q + (f_o - f_m)\\ &= (64-50)Q + (\underbrace{10000}_{\text{unavoidable}} - 10000) = 14Q > 0 \textit{ for all } Q > 0.\end{aligned}$$

This cost difference function says that if the $10,000 of fixed costs are unavoidable, then the proposed outsourcing contract would result in increased costs of $14 per unit *for any production volume*. So the firm would not minimize costs (nor maximize profits) by choosing to outsource production of the generators if the fixed costs were unavoidable.

Multi-period effects. The two solutions to the firm's cost minimization problem shown above ignore potential multi-period effects of outsourcing production including changes in product quality, the future inability to attract production workers after outsourcing, future changes in outsourcing price per unit, etc. Ordinarily outsourcing decisions are considered very carefully because of such multi-period effects.

The only fundamental difference between the analysis presented above and the analysis of a capital asset investment decision is the *cause* of the expected changes in the firm's cost function: In the outsourcing decision the cost function changes because of moving production to a supplier outside the firm, while in a capital asset investment decision the cost function changes because the production method (or at least the details of the production method) changes as a result of the investment. So it follows that the capital asset investment decision can be analyzed in precisely the same way, with only the notation (i.e., the subscripts) changing:

$$\begin{aligned}\Delta C_I(Q) &= C_I(Q) - C_{existing}(Q)\\ &= (v_I - v_{existing})Q + (f_I - f_{existing})\\ &= \Delta v_I Q + \Delta f_I\end{aligned}$$

The indifference point between investing in the capital assets (denoted by subscript *I*) can be found using the formula:

$$\underbrace{\Delta C_I(Q)}_{0} = \Delta v_I Q + \Delta f_I \quad \Rightarrow \quad Q_{\text{indifference}} = \Delta f_I / \Delta v_I,$$

while being careful that the changes in variable and fixed costs are calculated as $\Delta v_I = v_I - v_{existing}$ and $\Delta f_I = f_I - f_{existing}$.

As with the potential multi-period effects of an outsourcing decision, the multi-period effects of a capital asset investment decision can substantially alter the analysis and decision. We will study simple multi-period capital asset investment decisions in later chapters.

Profit maximization

Common profit maximization decisions often framed as single-period decisions can be categorized as:

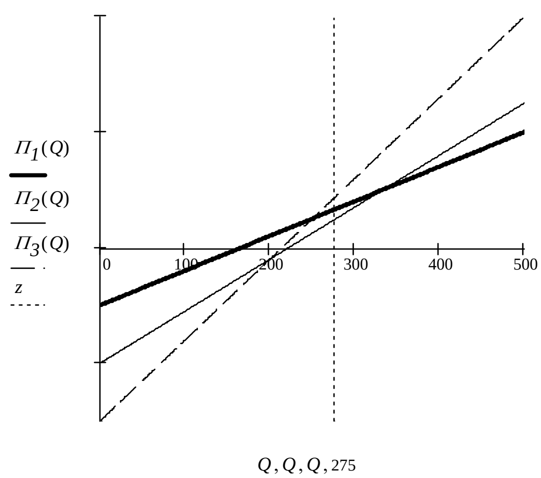
- Special order acceptance decisions
- Sell-or-process-further decisions
- Product selection decisions

Similar to cost minimization decisions using linear cost function, profit maximization decisions using linear profit functions are most often framed as choosing the profit-maximizing expected profit function from a set of profit functions:

$$\left\{ \begin{array}{c} \pi_1(Q) = (p_1 - v_1)Q - f_1 \\ \pi_2(Q) = (p_2 - v_2)Q - f_2 \\ \vdots \\ \pi_k(Q) = (p_k - v_k)Q - f_k \end{array} \right\}.$$

It is always important to remember that any profit function is comprised of one or more variables, and one or more parameters. The parameters must either be *known* (e.g., unit sales price *p* is known if the firm has a contract to sell its product for *p* per unit) or estimated (e.g., we studied how to estimate cost function parameters in Unit 2).

To see the nature of the decision a bit more clearly consider the following graph of three expected profit functions:



As can be seen at any given expected unit sales volume Q, expected profits generally differ under the three profit functions. For example, notice that at unit sales volume of 275,

$$\pi_3(Q = 275) > \pi_1(Q = 275) > \pi_2(Q = 275).$$

Also notice that at different unit sales volume amounts the expected profit-maximizing choice of profit function differs.

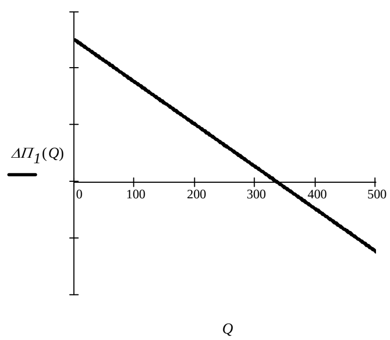
We have so far ignored the effects of *income taxes* on profit maximization decisions since, except in smaller firms, they tend to have minimal effect on the decision itself (though a large effect on the resulting profits). We will consider the effects of income taxes in somewhat more detail when we study multi-period decisions in later chapters.

Single product profit maximization decisions. Also similar to linear cost minimization, linear profit maximization decisions can sometimes be framed within a profit difference function:

$$\begin{aligned}\Delta\pi_1(Q) &= \pi_1(Q) - \pi_2(Q) \\ &= [(p_1 - v_1)Q - f_1] - [(p_2 - v_2)Q - f_2] \\ &= (cm_1 Q - f_1) - (cm_2 Q - f_2) \\ &= (cm_1 - cm_2)Q - (f_1 - f_2) \\ &= \Delta cm_1 Q - \Delta f_1\end{aligned}$$

where $cm = p - v$ denotes *contribution margin per unit* (a simplification often made by managers and accountants). Defining a profit difference function usually makes sense only when the firm is considering different pricing and cost functions for a *single product*; otherwise the *Q* cannot reasonably be factored as $(cm_1 - cm_2)Q - (f_1 - f_2)$ in the above equations.

Once a profit maximization *problem* (the solution to which would be considered the *decision*) for a single product has been framed within a function $\Delta\pi_1(Q) = \Delta cm_1 Q - \Delta f_1$, then (*i*) the slope Δcm_1 of the profit difference function, (*ii*) the indifference point Q_{indiff} at which $\Delta\pi_1(Q_{\text{indiff}}) = 0$, and (*iii*) the respective ranges of *Q* where either π_1 or π_2 maximizes profits. For example, the following graph of a profit difference function $\Delta\pi_1(Q) = \Delta cm_1 Q - \Delta f_1$,



shows that (*i*) the function is negatively sloping, (*ii*) the indifference point between π_1 and π_2 is somewhere around $Q \approx (325, 350)$, and (*iii*) $\pi_1 > \pi_2$ when $Q < Q_{\text{indifference}}$ and $\pi_1 < \pi_2$ when $Q > Q_{\text{indifference}}$. So the profit-maximizing choice depends on whether expected unit sales $\bar{Q} < Q_{\text{indiff}}$ or $\bar{Q} > Q_{\text{indiff}}$.

Multiple product decisions. Sometime managers simply need what is known as "cost-volume-profit" (CVP) information aggregate for a large number of different products for use in general business planning. Consider a firm with *N* different products and the related *N* different profit functions:

$$\left\{ \begin{array}{c} \pi_1(Q) = (p_1 - v_1)Q - f_1 \\ \pi_2(Q) = (p_2 - v_2)Q - f_2 \\ \vdots \\ \pi_N(Q) = (p_N - v_N)Q - f_N \end{array} \right\}.$$

How could the firm formulate an *aggregate profit function* that would provide information on the relationships between the firm's aggregate costs, product volumes, and profits? Notice that the firm's aggregate profits can be written as

$$\begin{aligned}\Pi(Q_1,...,Q_N) &= \pi_1(Q_1)+...+\pi_N(Q_N)\\ &= \underbrace{(cm_1Q_1+...+cm_NQ_N)}_{CM\text{ = total contribution margin}} - \underbrace{(f_1+...+f_N+f_{\text{other}})}_{F\text{ = total fixed costs}}\cdot\end{aligned}$$

Although there is no simple way to write an aggregate profit function in terms of product units, it is possible to write aggregate profits as a function of *total revenues,* $R = R_1 + ... + R_N = p_1Q_1 + ... + p_NQ_N$:

$$\Pi(R) = \left(\frac{CM}{R}\right)R - F = \underbrace{CMR}_{\substack{\text{average}\\ \text{contribution}\\ \text{margin}\\ \text{ratio}}} \cdot R - F\ ,$$

where *CMR* is called the *contribution margin ratio*. It can be shown that *CMR* is actually a weighted average of the firm's contribution margin per unit for each of the *N* individual products, where the weighting factors are the relative revenues of product *n* (R_n) to total revenues (R):

$$CMR = \left(p_1 - v_1\right)\frac{R_1}{R} + ... + \left(p_N - v_N\right)\frac{R_N}{R}\,.$$

The expression suggests a fundamental problem with the aggregate profit function $\Pi(R) = CMR \cdot R - F$: *It assumes that as revenues change, the relative weights* $R_1/R,...,R_N/R$ *remain constant*. Unfortunately, this is rarely the case in the real world. And yet managers can and do use such aggregate profit functions to explore what might happen to the firm's profits if revenues, contribution margins, and fixed costs change by some amount. A better method would be to explore such changes directly using an aggregate profit function stated in terms of the individual product units, $\Pi(Q_1,...,Q_N) = \pi_1(Q_1) + ... + \pi_N(Q_N)$, which is not particularly difficult using quantitative spreadsheet software (e.g., Microsoft Excel).

Special order decisions often arise as a result of a firm considering (*i*) selling an existing product at a price below its normal selling price, or (*ii*) making a product based on a customer's specifications and selling it at a contractually agreed-upon price. There is nothing particularly unique about the analysis of special order decisions in the sense that the firm simply wants to know whether profits will be increased (and, so, maximized) by accepting the special order or not. Algebraically, the firm wants to determine whether the following expected profit difference equation is greater than zero:

$$\Delta\pi_{\text{accept}} = \pi_{\text{accept}} - \pi > 0\,,$$

where π_{accept} denotes expected profits if the special order is accepted, and π denotes expected profits if the special order is not accepted. The problem might seem unique, however, since $\Delta\pi_{\text{accept}}$ simplifies to

$$\begin{aligned}\Delta\pi_{\text{accept}}(Q_{\text{so}}) &= \pi_{\text{accept}}(Q_{\text{so}}) - \pi(Q)\\ &= \underbrace{[\pi_{\text{so}}(Q_{\text{so}}) + \pi(Q)]}_{\pi_{\text{accept}}(Q_{\text{so}})} - \pi(Q)\\ &= (p_{\text{so}} - v_{\text{so}})Q_{\text{so}} - f_{\text{so}} > 0\end{aligned}$$

where the subscript *so* denotes the parameters and variable expected to result directly from the *special order*; so the profit function for the existing product(s), per se, need not be considered. This simply says that if the *marginal profit* from accepting the special order is positive, then the firm should accept the order. Perhaps the only difficult aspect of analyzing such decisions is in determining the amount of fixed costs that *specifically result* from accepting the special order, f_{so}.

Since special orders tend to be priced lower than ordinary products and are generally *negotiated*, another somewhat unique aspect of analyzing special order decisions is *regarding the profit difference as a function of the price per unit of the special order* P_{so}, rather than unit order quantity q_{so}:

$$\Delta\pi_{\text{accept}}(P_{\text{so}}) = (P_{\text{so}} - v_{\text{so}})q_{\text{so}} - f_{\text{so}}.$$

This often makes sense since unit order quantity is often fixed by the prospective customer. The firm then would be interested in the unit selling price at which it becomes profitable to accept the special order (i.e., the break-even unit price for the special order, $P_{\text{so}}^{\text{BE}}$):

$$\underbrace{\Delta\pi_{\text{accept}}(P_{\text{so}})}_{0} = (P_{\text{so}} - v_{\text{so}})q_{\text{so}} - f_{\text{so}} \quad\Rightarrow\quad P_{\text{so}}^{\text{BE}} = \underbrace{v_{\text{so}} + \frac{f_{\text{so}}}{q_{\text{so}}}}_{\text{average cost per unit}}.$$

Since the function has a positive slope in P_{so} it follows that the firm would want to accept the special order only if $P_{\text{so}} \geq P_{\text{so}}^{\text{BE}}$.

<u>Sell-or-process-further decisions</u> arise from the possibility of further processing an existing product to make another product (e.g., crude oil can be sold or refined into one or more products like gasoline, motor oil, other lubricants, etc.). Similar to special order decision analysis, there is nothing particularly unique about analyzing sell-or-process decisions since they again simply require determining whether the marginal profit from further processing is positive or not:

$$\begin{aligned}\Delta\pi_{\text{proc}}(Q) &= \pi_{\text{proc}}(Q) - \pi_{\text{sell}}(Q)\\ &= \underbrace{(cm_{\text{proc}}Q - f_{\text{proc}})}_{\pi_{\text{proc}}(Q)} - \underbrace{(cm_{\text{sell}}Q - f_{\text{sell}})}_{\pi_{\text{sell}}(Q)}\\ &= (cm_{\text{proc}} - cm_{\text{sell}})Q - (f_{\text{proc}} - f_{\text{sell}})\\ &= \Delta cm_{\text{proc}}Q - \underbrace{\Delta f_{\text{proc}}}_{0} > 0\end{aligned}$$

where Q is *stated in terms of the product that can be sold before further processing* (e.g., crude oil rather than any product refined from the oil), and in general $\Delta f_{\text{proc}} = 0$ if the firm has the capacity to further process all Q units. It is important to recognize that $cm_{\text{proc}} = p_{\text{proc}} - v_{\text{proc}}$ represents the contribution margin per unit from the further-processed unit, *not* the marginal change due to the further processing $cm_{\text{proc}} = \Delta p_{\text{proc}} - \Delta v_{\text{proc}}$.

Since sell-or-process decisions tend to depend on the expected market price of the further-processed-product, which is often uncertain, and perhaps volatile, firms are often interested in writing the profit difference function in terms of the expected market price for the further-processed-product P_{proc}:

$$\begin{aligned}\Delta\pi_{\text{proc}}(P_{\text{proc}}) &= \Delta cm_{\text{proc}}(P_{\text{proc}}) - \Delta f_{\text{proc}} \\ &= \underbrace{[(P_{\text{proc}} - v_{\text{proc}})Q - (P_{\text{sell}} - v_{\text{sell}})Q]}_{\Delta cm_{\text{proc}}(P_{\text{proc}})} - \Delta f_{\text{proc}}.\end{aligned}$$

Similar to the special order decision analysis, the expected price at which it becomes profitable to further process the product can be found as

$$\underbrace{\Delta\pi_{\text{proc}}(P_{\text{proc}})}_{0} = [(P_{\text{proc}} - v_{\text{proc}}) - (p_{\text{sell}} - v_{\text{sell}})]q - \Delta f_{\text{proc}} \quad \Rightarrow \quad P_{\text{proc}}^{\text{BE}} = \underbrace{v_{\text{proc}} + cm_{\text{sell}} + \frac{\Delta f_{\text{proc}}}{q}}_{\substack{\text{average cost to make} \\ \text{further-processed unit} \\ \text{incl cost of existing} \\ \text{product}}};$$

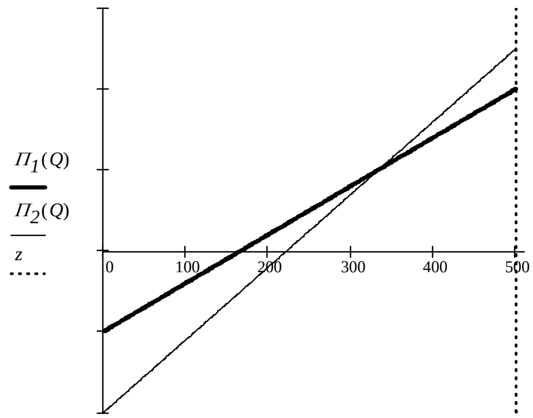
then since the function has a positive slope in P_{proc} it follows that the firm would want to further process the existing product only if the expected unit price exceeds the break-even price, $P_{\text{so}} \geq P_{\text{so}}^{\text{BE}}$.

Product selection decisions and constraints. As we have discussed linear functions must generally be constrained to obtain meaningful solutions to maximization problems. In the context of simple profit maximization problems this usually means obtaining estimates of the total unit production or sales volume possible for the firm. When a firm is selecting between two or more products to make and sell, it is also important (perhaps more important) to consider production and sales constraints since different products generally have different contribution margins and different production and sales constraints. Algebraically the constraints generally take the form

$$\begin{aligned} Q_1 &\leq Q_{1\max} \\ &\vdots \\ Q_N &\leq Q_{N\max} \\ Q_1 + \ldots + Q_N &\leq Q_{total\max} \end{aligned}$$

indicating that each product has a maximum production or sales volume, and the firm has an overall (usually production) volume. Though constrained maximization of sets of linear functions can be quite complex (and beyond the scope of the course), we will focus only on simple cases. We will however derive a general approach to product selection decisions by studying profit maximization through product selection with varying types of constraints.

Overall production constraint. Consider a production selection decision where the firm can make and sell any combination of two products but only 500 units can be produced (i.e., $Q_1 + Q_2 \leq 500$) as represented in the following graph:

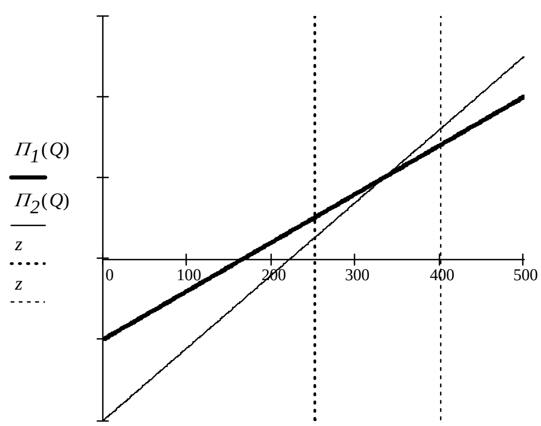


$Q, Q, 500$

In this case it is easy to see that,

> *If the expected sales volume for each product exceeded the indifference point between the two functions,* $\bar{Q}_1 > Q_{\text{indiff}}$ *and* $\bar{Q}_2 > Q_{\text{indiff}}$ (about 325-350 in the graph above), *the profit-maximizing choice would be the product with the higher contribution margin*.

Individual product constraints and a nonbinding overall production constraint. Consider again the same basic production selection decision where the firm can make and sell any combination of two products, but in this case (*i*) 1000 units of either product can be made in combination ($Q_1 + Q_2 \leq 1000$) and (*ii*) a maximum of 250 units of Product 1 ($Q_1 < 250$) and a maximum of 400 units of Product 2 ($Q_2 < 400$) can be sold, as shown in the graph:



$Q, Q, 250, 400$

How could the firm maximize profits given the expected profit functions and the three constraints? Since

$$\underbrace{Q_{1\max}}_{250} + \underbrace{Q_{2\max}}_{400} < \underbrace{Q_{\max\text{ total}}}_{1000}$$

the firm maximizes profits by making and selling $Q_{1\max} + Q_{2\max} = 250 + 400 = 650$; where, for example, $(Q_1, Q_2) = (250, 400)$ is called the profit-maximizing *production plan*.

Binding total production and individual product constraints. Following the last paragraph, if (*i*) the overall production constraint is reduced so the firm can only make a total of 500 units of either product (in any combination, $Q_1 + Q_2 \leq 500$) and (*ii*) a maximum of 250 units of Product 1 ($Q_1 \leq 250$) and a maximum of 400 units of Product 2 ($Q_2 \leq 400$) can be sold, then how does the optimal production plan change? The answer is fairly intuitive: The firm would maximize profits by making and selling as many units as possible of its "most profitable" product, and then use any remaining capacity to make the next most profitable product, and so on. It is simply necessary to define what is meant by *most profitable*: The *most profitable product* is the product with the highest contribution margin per unit.

So in this case we can determine the most profitable product simply by looking at the graph from the previous paragraph and observing the slope of the Product 2 profit function is higher than that of Product 1. It follows that the firm would maximize profits by making and selling $Q_1 = (Q_{1\max} = 250)$, and then use its remaining capacity to make and sell $Q_2 = Q_{total\max} - Q_{1\max} = 500 - 250 = 250$ units of Product 2.

General approach to product selection with constraints. We have in fact derived a general approach to maximizing sets of linear profit functions in the presence of constraints, which can be summarized as

(1) Identify the firm's total production constraint ($Q_1 + \ldots + Q_N \leq Q_{total\max}$);

(2) Identify constraints on the firm's individual products ($Q_1 \leq Q_{1\max}, \ldots, Q_N \leq Q_{N\max}$);

(3) Rank products from highest-to-lowest contribution margin; and then

(4) Assign available production capacity to products in their highest-to-lowest rank order such that all available capacity is used.

This approach is quite general and is appropriate even for simple cases where the constraints are "large" (i.e., too large to think of them as constraints in the normal sense of the word). The generality of the approach can be seen by drawing graphs of profit functions and reviewing them in conjunction with the logic presented in this subsection.