7. Four problems with nonlinear objective functions solved by linear methods.

1. Constrained Games. Find x_j for $j = 1, \ldots, n$ to maximize

$$\min_{1 \le i \le p} \left[\sum_{j=1}^{n} c_{ij} x_j \right] \tag{1}$$

subject to the constraints as in the general maximum problem

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\le b_i \qquad \text{for } i = 1, \ldots, k \\ \sum_{j=1}^{n} a_{ij} x_j &= b_i \qquad \text{for } i = k+1, \ldots, n \end{aligned} \tag{2}$$

and

$$\begin{aligned} x_j &\ge 0 \qquad \text{for } j = 1, \ldots, \ell \\ (x_j &\text{ unrestricted} \qquad \text{for } j = \ell + 1, \ldots, n) \end{aligned} \tag{3}$$

This problem may be considered as a generalization of the matrix game problem from player I's viewpoint, and may be transformed in a similar manner into a general linear program, as follows. Add λ to the list of unrestricted variables, subject to the constraints

$$\lambda - \sum_{j=1}^{n} c_{ij} x_j \le 0 \qquad \text{for } i = 1, \ldots, p \tag{4}$$

The problem becomes: maximize λ subject to the constraints (2), (3) and (4).

Example 1. The General Production Planning Problem (See Zukhovitskiy and Avdeyeva, "Linear and Convex Programming", (1966) W. B. Saunders pg. 93) There are n activities, $A_1, \ldots, A_n$, a company may employ using the available supplies of m resources, $R_1, \ldots, R_m$. Let b_i be the available supply of R_i and let a_{ij} be the amount of R_i used in operating A_j at unit intensity. Each activity may produce some or all of the p different parts that are used to build a complete product (say, a machine). Each product consists of N_1 parts #1,... ,N_p parts $\#p$. Let c_{ij} denote the number of parts $\#i$ produced in operating A_j at unit intensity. The problem is to choose activity intensities to maximize the number of complete products built.

Let x_j be the intensity at which A_j is operated, $j = 1, \ldots, n$. Such a choice of intensities produces $\sum_{j=1}^{n} c_{ij} x_j$ parts $\#i$, which would be required in building $\sum_{j=1}^{n} c_{ij} x_j / N_i$ complete products. Therefore, such an intensity selection may be used to build

$$\min_{1 \le i \le p} \left[\sum_{j=1}^{n} c_{ij} x_j / N_i \right] \tag{5}$$

complete products. The amount of R_i used in this intensity selection must be no greater than b_i,

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \qquad \text{for } i = 1, \ldots, m \tag{6}$$

and we cannot operate at negative intensity

$$x_j \ge 0 \qquad \text{for } j = 1, \ldots, n \tag{7}$$

We are required to maximize (5) subject to (6) and (7). When $p = 1$, this reduces to the Activity Analysis Problem.

Exercise 1. In the general production planning problem, suppose there are 3 activities, 1 resource and 3 parts. Let $b_1 = 12$, $a_{11} = 2$, $a_{12} = 3$ and $a_{13} = 4$, $N_1 = 2$, $N_2 = 1$, and $N_3 = 1$, and let the c_{ij} be as given in the matrix below. (a) Set up the simplex tableau of the associated linear program. (b) Solve. (Ans. $x_1 = x_2 = x_3 = 4/3$, value = 8.)

$$\mathbf{C} = \begin{matrix} \\ \text{part \#1} \\ \text{part \#2} \\ \text{part \#3} \end{matrix} \begin{matrix} \begin{matrix} A_1 & A_2 & A_3 \end{matrix} \\ \begin{pmatrix} 2 & 4 & 6 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} \end{matrix}$$

2. Minimizing the sum of absolute values. Find y_i for $i = 1, \ldots, m$, to minimize

$$\sum_{j=1}^{p} \left| \sum_{i=1}^{m} y_i b_{ij} - b_j \right| \tag{8}$$

subject to the constraints as in the general minimum problem

$$\begin{aligned} \sum_{i=1}^{m} y_i a_{ij} &\ge c_j \qquad \text{for } j = 1, \ldots, \ell \\ \sum_{i=1}^{m} y_i a_{ij} &= c_j \qquad \text{for } j = \ell + 1, \ldots, n \end{aligned} \tag{9}$$

and

$$\begin{aligned} y_i &\ge 0 \qquad \text{for } i = 1, \ldots, k \\ (y_i \text{ unrestricted} & \qquad \text{for } i = k + 1, \ldots, m.) \end{aligned} \tag{10}$$

To transform this problem to a linear program, add p more variables, $y_{m+1}, \ldots, y_{m+p}$, where y_{m+j} is to be an upper bound of the jth term of (8), and try to minimize $\sum_{j=1}^{p} y_{m+j}$. The constraints

$$\left| \sum_{i=1}^{m} y_i b_{ij} - b_j \right| \le y_{m+j} \qquad \text{for } j = 1, \ldots, p$$

are equivalent to the $2p$ linear constraints

$$\begin{aligned} \sum_{i=1}^{m} y_i b_{ij} - b_j &\le y_{m+j} \qquad \text{for } j = 1, \ldots, p \\ -\sum_{i=1}^{m} y_i b_{ij} + b_j &\le y_{m+j} \qquad \text{for } j = 1, \ldots, p \end{aligned} \tag{11}$$

The problem becomes minimize $\sum_{j=1}^{p} y_{m+j}$ subject to the constraints (9), (10) and (11). In this problem, the constraints (11) imply that $y_{m+1}, \ldots, y_{m+p}$ are nonnegative, so it does not matter whether these variables are restricted to be nonnegative or not. Computations are generally easier if we leave them unrestricted.

Example 2. It is desired to find an m-dimensional vector, $\boldsymbol{y}$, whose average distance to p given hyperplanes

$$\sum_{i=1}^{m} y_i b_{ij} = b_j \qquad \text{for } j = 1, \ldots, p$$

is a minimum. To find the (perpendicular) distance from a point $(y_1^0, \ldots, y_m^0)$ to the plane $\sum_{i=1}^{m} y_i b_{ij} = b_j$, we normalize this equation by dividing both sides by $d_j = (\sum_{i=1}^{m} b_{ij}^2)^{1/2}$. The distance is then $|\sum_{i=1}^{m} y_i^0 b'_{ij} - b'_j|$, where $b'_{ij} = b_{ij}/d_j$ and $b'_j = b_j/d_j$. Therefore, we are searching for $y_1, \ldots, y_m$ to minimize

$$\frac{1}{p} \sum_{j=1}^{p} \left| \sum_{i=1}^{m} y_i b'_{ij} - b'_j \right|.$$

There are no constraints of the form (9) or (10) to the problem as stated.

Exercise 2. Consider the problem of finding y_1 and y_2 to minimize $|y_1 + y_2 - 1| + |2y_1 - y_2 + 1| + |y_1 - y_2 - 2|$ subject to the constraints $y_1 \ge 0$ and $y_2 \ge 0$. (a) Set up the simplex tableau of the associated linear program. (b) Solve. (Ans. $(y_1, y_2) = (0, 1)$, value $= 3$.)

3. Minimizing the maximum of absolute values. Find $y_1, \ldots, y_m$ to minimize

$$\max_{1 \le j \le p} \left| \sum_{i=1}^{m} y_i b_{ij} - b_j \right| \tag{12}$$

subject to the general constraints (9) and (10). This objective function combines features of the objective functions of 1. and 2. A similar combination of methods transforms this problem to a linear program. Add μ to the list of unrestricted variables subject to the constraints

$$\left| \sum_{i=1}^{m} y_i b_{ij} - b_j \right| \le \mu \qquad \text{for } j = 1, \ldots, p$$

and try to minimize μ. These p constraints are equivalent to the following $2p$ linear constraints

$$\begin{aligned} \sum_{i=1}^{m} y_i b_{ij} - b_j &\leq \mu \qquad \text{for } j = 1, \ldots, p \\ -\sum_{i=1}^{m} y_i b_{ij} + b_j &\leq \mu \qquad \text{for } j = 1, \ldots, p \end{aligned} \tag{13}$$

The problem becomes: minimize μ subject to (9), (10) and (13).

Example 3. Chebyshev Approximation. Given a set of p linear affine functions in m unknowns,

$$\psi_j(y_1, \ldots, y_m) = \sum_{i=1}^{m} y_i b_{ij} - b_j \qquad \text{for } j = 1, \ldots, p, \tag{14}$$

find a point, $(y_1^0, \ldots, y_m^0)$, for which the maximum deviation (12) is a minimum. Such a point is called a Chebyshev point for the system (14), and is in some sense a substitute for the notion of a solution of the system (14) when the system is inconsistent. Another substitute would be the point that minimizes the total deviation (8). If the functions (14) are normalized so that $\sum_{i=1}^{m} b_{ij}^2 = 1$ for all j, then the maximum deviation (12) becomes the maximum distance of a point $\boldsymbol{y}$ to the planes

$$\sum_{i=1}^{m} y_i b_{ij} = b_j \qquad \text{for } j = 1, \ldots, p.$$

Without this normalization, one can think of the maximum deviation as a "weighted" maximum distance. (See Zukhovitskiy and Avdeyeva, pg. 191, or Stiefel "Note on Jordan Elimination, Linear Programming, and Tchebycheff Approximation" *Numerische Mathematik* **2** (1960), 1-17.

Exercise 3. Find a Chebyshev point (*unnormalized*) for the system

$$\begin{aligned} \psi_1(y_1, y_2) &= y_1 + y_2 - 1 \\ \psi_2(y_1, y_2) &= 2y_1 - y_2 + 1 \\ \psi_3(y_1, y_2) &= y_1 - y_2 - 2 \end{aligned}$$

by (a) setting up the associated linear program, and (b) solving. (Ans. $y_1 = 0$, $y_2 = -1/2$, value $= 3/2$.)

4. Linear Fractional Programming. (Charnes and Cooper, "Programming with linear fractional functionals", *Naval Research Logistics Quarterly* **9** (1962), 181-186.) Find $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathsf{T}}$ to maximize

$$\frac{\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \alpha}{\boldsymbol{d}^{\mathsf{T}}\boldsymbol{x} + \beta} \tag{15}$$

subject to the general constraints (2) and (3). Here, $\boldsymbol{c}$ and $\boldsymbol{d}$ are n-dimensional vectors and α and β are real numbers. To avoid technical difficulties we make two assumptions: that the constraint set is nonempty and bounded, and that the denominator $\boldsymbol{d}^T\boldsymbol{x} + \beta$ is strictly positive throughout the constraint set.

Note that the objective function remains unchanged upon multiplication of numerator and denominator by any number $t > 0$. This suggests holding the denominator fixed say $(\boldsymbol{d}^T\boldsymbol{x} + \beta)t = 1$, and trying to maximize $\boldsymbol{c}^T\boldsymbol{x}t + \alpha t$. With the change of variable $\boldsymbol{z} = \boldsymbol{x}t$, this becomes embedded in the following linear program. Find $\boldsymbol{z} = (z_1, \ldots, z_n)$ and t to maximize

$$\boldsymbol{c}^T\boldsymbol{z} + \alpha t$$

subject to the constraints

$$\begin{aligned} &\boldsymbol{d}^T\boldsymbol{z} + \beta t = 1 \\ &\sum_{j=1}^{n} a_{ij} z_j \le b_i t \qquad \text{for } i = 1, \ldots, k \\ &\sum_{j=1}^{n} a_{ij} z_j = b_i t \qquad \text{for } i = k+1, \ldots, m \end{aligned}$$

and

$$\begin{aligned} t &\ge 0 \\ z_j &\ge 0 \qquad \text{for } j = 1, \ldots, \ell \\ z_j &\text{ unrestricted} \qquad \text{for } j = \ell + 1, \ldots, n. \end{aligned}$$

Every value achievable by a feasible $\boldsymbol{x}$ in the original problem is achievable by a feasible $(\boldsymbol{z}, t)$ in the linear program, by letting $t = (\boldsymbol{d}^T\boldsymbol{x} + \beta)^{-1}$ and $\boldsymbol{z} = \boldsymbol{x}t$. Conversely, every value achievable by a feasible $(\boldsymbol{z}, t)$ with $t > 0$ in the linear program is achievable by a feasible $\boldsymbol{x}$ in the original problem by letting $\boldsymbol{x} = \boldsymbol{z}/t$. But t cannot be equal to zero for any feasible $(\boldsymbol{z}, t)$ of the linear program, since if $(\boldsymbol{z}, 0)$ were feasible, $\boldsymbol{x} + \delta\boldsymbol{z}$ would be feasible for the original program for all $\delta > 0$ and any feasible $\boldsymbol{x}$, so that the constraint set would either be empty or unbounded. (One may show t cannot be negative either, so that it may be taken as one of the unrestricted variables if desired.) Hence, a solution of the linear program always leads to a solution of the original problem.

Example 4. Activity analysis to maximize rate of return. There are n activities $A_1, \ldots, A_n$ a company may employ using the available supply of m resources $R_1, \ldots, R_m$. Let b_i be the available supply of R_i and let a_{ij} be the amount of R_i used in operating A_j at unit intensity. Let c_j be the net return to the company for Operating A_j at unit intensity, and let d_j be the time consumed in operating A_j at unit intensity. Certain other activities not involving $R_1, \ldots, R_m$ are required of the company and yield net return α at a time consumption of β. The problem is to maximize the rate of return (15) subject to the restrictions $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$ and $\boldsymbol{x} \ge \boldsymbol{0}$. We note that the constraint set is nonempty if $\boldsymbol{b} \ge \boldsymbol{0}$, that it is generally bounded (for example if $a_{ij} > 0$ for all i and j), and that $\boldsymbol{d}^T\boldsymbol{x} + \beta$ is positive on the constraint set if $\boldsymbol{d} \ge \boldsymbol{0}$ and $\beta > 0$.