1. For any $\mathbf{x} \in \mathbb{C}^n$, prove

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_1 \leq n\,\|\mathbf{x}\|_\infty \ ,$$
$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\,\|\mathbf{x}\|_\infty \ ,$$
$$\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\,\|\mathbf{x}\|_2 \ .$$

2. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be invertible. Prove that

$$\frac{1}{\text{lub}(\mathsf{A}^{-1})} = \min_{\mathbf{y}\in\mathbb{C}^n_\star} \frac{\|\mathsf{A}\mathbf{y}\|}{\|\mathbf{y}\|} \ .$$

3. Suppose that $\mathsf{A}, \mathsf{B} \in \mathbb{C}^{n\times n}$ and A is non-singular and B is singular. Let $\|\cdot\|$ be an subordinate matrix norm. Prove that

$$\frac{1}{\text{cond}\,(\mathsf{A})} \leq \frac{\|\mathsf{A}-\mathsf{B}\|}{\|\mathsf{A}\|} \ ,$$

where $\text{cond}(\mathsf{A}) = \|\mathsf{A}\| \cdot \|\mathsf{A}^{-1}\|$.

Note: This formula is useful in a couple of ways. First, it says that if A is close in norm to a singular matrix B, then $\text{cond}(\mathsf{A})$ will be very large. Thus, nearly singular matrices are ill-conditioned. Second, this formula gives an upper bound on $\text{cond}(\mathsf{A})^{-1}$.

4. Let $\mathsf{A} \in \mathbb{C}^{n\times n}$ be invertible. Prove that

$$\frac{1}{\text{cond}_2\,(\mathsf{A})} = \inf_{\det(\mathsf{B})=0} \frac{\|\mathsf{A}-\mathsf{B}\|_2}{\|\mathsf{A}\|_2} \ .$$

5. Suppose

$$\mathsf{A} = \begin{bmatrix} 1.0000 & 2.0000 \\ 1.0001 & 2.0000 \end{bmatrix} \ .$$

   (a) Calculate $\text{cond}_1\,(\mathsf{A}) := \|\mathsf{A}\|_1 \cdot \|\mathsf{A}^{-1}\|_1$ and $\text{cond}_\infty\,(\mathsf{A}) := \|\mathsf{A}\|_\infty \cdot \|\mathsf{A}^{-1}\|_\infty$.

   (b) Use the result of problem 1 to obtain upper bounds on $\text{cond}_1(\mathsf{A})^{-1}$ and also on $\text{cond}_\infty(\mathsf{A})^{-1}$.

(c) Suppose that you wish to solve $\mathsf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 3.0000 \\ 3.0001 \end{bmatrix}$. Instead of $\mathbf{x}$ you obtain the approximation $\mathbf{x}' = \mathbf{x} + \delta\mathbf{x} = \begin{bmatrix} 0.0000 \\ 1.5000 \end{bmatrix}$. For this approximation you discover $\mathbf{b}' = \mathbf{b} + \delta\mathbf{b} = \begin{bmatrix} 3.0000 \\ 3.0000 \end{bmatrix}$, where $\mathsf{A}\mathbf{x}' = \mathbf{b}'$. Calculate $\|\delta\mathbf{x}\|_1 / \|\mathbf{x}\|_1$ exactly. (You will need the exact solution, of course). Then use the general estimate

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \operatorname{cond}(\mathsf{A}) \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

to obtain an upper bound for $\|\delta\mathbf{x}\|_1 / \|\mathbf{x}\|_1$. How good is $\|\delta\mathbf{b}\|_1 / \|\mathbf{b}\|_1$ as indicator of the size of $\|\delta\mathbf{x}\|_1 / \|\mathbf{x}\|_1$.

6. Suppose

$$\mathsf{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

where a and b are real numbers. Show that the subordinate matrix norms satisfy $\|\mathsf{A}\|_1 = \|\mathsf{A}\|_2 = \|\mathsf{A}\|_\infty$.

7. Suppose that

$$\mathsf{A} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

where a and b are real numbers. Show $\|\mathsf{A}\|_2 = (a^2 + b^2)^{1/2}$.

8. Show that if λ is an eigenvalue of $\mathsf{A}^H\mathsf{A}$, where $\mathsf{A} \in \mathbb{C}^{n\times n}$, then

$$0 \leq \lambda \leq \|\mathsf{A}^H\| \, \|\mathsf{A}\| .$$

9. Suppose that $\mathsf{A} \in \mathbb{C}^{n\times n}$ is invertible. Show that

$$\operatorname{cond}_2(\mathsf{A}) = \sqrt{\frac{\lambda_n}{\lambda_1}},$$

where λ_n is the largest eigenvalue of $\mathsf{B} := \mathsf{A}^T\mathsf{A}$, and λ_1 is the smallest eigenvalue of B.

10. Suppose that $\mathsf{A} \in \mathbb{C}^{n\times n}$ is invertible. Use the last two problems to show that

$$\operatorname{cond}_2(\mathsf{A}) \leq \sqrt{\operatorname{cond}_1(\mathsf{A}) \operatorname{cond}_\infty(\mathsf{A})} .$$