UNIVERSITY OF NEWCASTLE UPON TYNE

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 1
2002/2003

MAS317

INSTABILITIES

Time allowed: **1 Hour 30 minutes**

Credit will be given for ALL answers to questions in Section A and the best TWO answers to questions in Section B.

No credit will be given for other answers, and students are strongly advised not to spend time producing answers for which they will receive no credit.

There are THREE questions in Section A and THREE questions in Section B. Marks allocated to each question are indicated. However, you are advised that marks indicate the relative weight of individual questions; they do not correspond directly to marks on the University scale.

SECTION A

A1. A thin layer of water of thickness $d = 0.5\,\text{cm}$ is heated from below. The bottom of the layer is held at temperature T_0 and the top of the layer is held at temperature $T_1 = 15^0 C$. The Rayleigh number of the system is defined as

$$R = \frac{\alpha\beta d^4 g}{\kappa\upsilon}$$

where $g = 998\ \text{cm/sec}^2$ is the acceleration due to gravity,

$\alpha = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{T=T_0} = 1.5x10^{-4}\ \ {}^\circ C^{-1}$ is the thermal expansion coefficient, $\beta = \Delta\text{T}/d$ is the temperature gradient,

$\Delta\text{T} = \text{T}_0 - \text{T}_1$, $\kappa = 1.4x10^{-3}\ \text{cm}^2/\text{sec}$ is the thermal diffusivity and $\upsilon = 1.1x10^{-2}\ \text{cm}^2/\text{sec}$ is the kinematic viscosity. Assume that the critical Rayleigh number for the appearance of convection is $R = R_c = 1708$.

(a) Determine the critical temperature T_0 at which convection appears.

(b) When water has temperature close to freezing (for example, a lake almost at $0^\circ C$ in the winter), the coefficient of thermal expansion α is negative, that is to say water expands when cooled rather than contract. Explain why in this situation convection takes place when the water is heated from above (for example, the sun heating the surface of the lake) rather than from below.

[10 marks]

A2. The population of birds on an island is described by the following model $\dfrac{dN}{dt} = rN - \alpha N^3$, (1)

where $r > 0$ and $\alpha > 0$ are parameters, t is time and N is the number of birds.

(a) Show that there exist two steady solutions $N_1 = 0$ and $N_2 = \sqrt{r/\alpha}$.

(b) Determine the linear stability of N_1 and N_2.

[12 marks]

A3. Consider an infinite layer of fluid held between two planes $z = 0$ and $z = \delta$ such that the temperature of the bottom boundary is T_0 and the temperature of the top boundary is T_1, with $T_0 > T_1$. The basic state is a state of no motion with a temperature gradient $(T_0 - T_1)/\delta$ across the layer. Assuming stress free boundary conditions and perturbations with dependence $e^{\sigma_n t + inax}$, the complex growth rate σ_n of the n^{th} mode is given by the equation

$$-\left(n^2\pi^2 + a^2\right)\left(\sigma_n + Pn^2\pi^2 + Pa^2\right)\left(\sigma_n + n^2\pi^2 + a^2\right) + a^2 PR = 0 \qquad (1)$$

where P is the Prandtl number, R the Rayleigh number, $a > 0$ the wavenumber, $\sigma_n = s_n + i\omega_n$ (with s_n and ω_n real) and n is an integer.

(a) Assume exchange of stability $(\omega_n = 0)$ and show that the marginal states (for which $s_n = 0$) are independent of P and are given by

$$R = \frac{\left(n^2\pi^2 + a^2\right)^3}{a^2}$$

(b) Minimize R with respect to n and with respect to a, and show that the critical Rayleigh number is $R_c = 27\pi^4/4$ and corresponds to the critical wavenumber $a_c = \pi/\sqrt{2}$

(c) Sketch the curve R *vs* a, identify the point (a_c, R_c) and state which regions of the graph correspond to stable (conduction) and unstable (convection) solutions.

[18 marks]

SECTION B

B4. Consider the following $\alpha\Omega$ model of the solar dynamo

$$\frac{\partial}{\partial t}B_x = \frac{\partial^2 B_x}{\partial y^2} + \alpha\frac{\partial B_z}{\partial y},$$

$$\frac{\partial}{\partial t}B_z = \frac{\partial^2 B_z}{\partial y^2} + \Omega B_x,$$

where $\underset{\sim}{B} = (B_x, 0, B_z)$ is the magnetic field and the constants $\alpha > 0$ and $\Omega > 0$ represent the alpha effect and the differential rotation respectively. In this model the cartesian directions x, y, z represent the radial, meridional and azimuthal directions in the sun. Assume that B_x and B_z depend only on y and t, and call $D = \alpha\Omega$ the dynamo number.

(a) Assume time dependence $e^{\sigma t + iky}$ for the magnetic field where k is the wavenumber and $\sigma = s + i\omega$ is the complex growth rate (s and ω real), and show that the linear stability is determined by the equation $\left(\sigma + k^2\right)^2 - ikD = 0$.

(b) By setting s=0, determine the marginal state of the dynamo and show that the critical value of the dynamo number for the magnetic field to grow is $D_c = 2k^3$

(c) Is the dynamo steady of oscillatory? If oscillatory, find the period $\tau = 2\pi / \omega$ of oscillations.

(d) What determines the minimum wavenumber k_c hence the minimum value of D_c?

[30 marks]

B5. Consider the following equation for $u > 0$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u^3 + \ell u \text{ where } \ell \text{ is a positive parameter.}$$

(1) Show that there are two steady (time independent) and uniform (space independent) solutions, $u_1 = 0$ and $u_2 = \sqrt{\ell}$.

(2) Perturb the basic state $u = u_1 = 0$ retaining only linear terms, and assume that the perturbations have dependence $e^{\sigma t - ikx}$.

(2.1) Determine for which values of k the basic state $u = u_1 = 0$ is stable or unstable.

(2.2) Is a perturbation of wavenumber $k = 2$ stable for $\ell = 1$?

(2.3) For which values of ℓ is the basic state $u = u_1 = 0$ stable, no matter what is the wavenumber k of the perturbation?

(3) Perturb the basic state $u = u_2 = \sqrt{\ell}$, again assuming dependence $e^{\sigma t - ikx}$ for the perturbations.

(3.1) Determine for which values of k the basic state $u = u_2 = \sqrt{\ell}$ is stable or unstable.

(3.2) For which value of ℓ is the basic state $u = u_2 = \sqrt{\ell}$ stable?

[30 marks]

B6. Consider the following equations

$$\frac{dx}{dt} = -x + y,$$

$$\frac{dy}{dt} = -xz + rx - y,$$

$$\frac{dz}{dt} = xy - bz,$$

where $b > 0$ and r is a parameter.

(a) Show that there are three steady solutions

i) $x_0 = y_0 = z_0 = 0$,

ii) $x_0 = y_0 = \sqrt{b(r-1)}$, $\quad z_0 = r - 1$,

iii) $x_0 = y_0 = -\sqrt{b(r-1)}$, $\quad z_0 = r - 1$.

(b) Perturb the basic state $x_0 = y_0 = z_0 = 0$ and show that the linearized equations for the perturbations x', y' and z' are

$$\frac{dx'}{dt} = -x' + y',$$

$$\frac{d}{dt}y' = rx' - y',$$

$$\frac{d}{dt}z' = -bz'.$$

(c) Assuming the dependence e^{st} for the perturbations, show that s is determined by $(s+1)^2 - r = 0$

(d) Determine for which values of r the basic state $x_0 = y_0 = z_0 = 0$ is stable and for which values it is unstable. Find the growth rate s for the unstable solution. Is the system stable or unstable for $r = 1$?

[30 marks]