Problem of Schwarz's Lemma

1. **Suppose f: D $\rightarrow$ ¢ satisfies Re $f(z) \geq 0$ for all z in D and suppose that f is analytic and non-constant.**

   (a) Show that Re $f(z) > 0$ for all z in D

   (b) By using an appropriate Mobius transformation, apply Schwarz's Lemma to prove that if f(0) = 1 then $|f(z)| \leq \frac{1+|z|}{1-|z|}$ for $|z| < 1$. What can be said if $f(0) \neq 1$?

   (c) Show if $f(0) = 1$, f also satisfies $|f(z)| \geq \frac{1-|z|}{1+|z|}$.

Nots: Re $f(z)$ indicates the residue of f(z)

Theorem, Lemma and Propositon:

Schwarz's Lemma:

Let D = {z: $|z| < 1$}, suppose f is analytic on D with

(1) $|f(z)| \leq 1$ for z in D

(2) $f(0) = 0$

Then $|f'(0)| <= 1$ and $|f(z)| \leq |z|$ for all z in D.

If $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some z not equal to zero, then there exists a constant c, $|c| = 1$, such that $f(w) = cw$ for all w in D.

Proposition:

If $|a| < 1$ then φ_a is a one-one map of D = {z: $|z| < 1$} onto itself; the inverse of φ_a is φ_{-a}. Furthermore, φ_a maps ∂D onto ∂D, $\varphi_a(a) = 0$, $\varphi_a'(0) = 1 - |a|^2$, and $\varphi_a'(a) = (1 - |a|^2)^{-1}$.

Theorem:

Let f: D $\rightarrow$ D be a one-one analytic map of D onto itself and suppose $f(z) = 0$. Then there is a complex number c with $|c| = 1$ such that $f = c\varphi_a$.

Definition:

A mapping of the form $S(z) = \frac{az+b}{cz+d}$ is called a linear fractional transformation. If a, b, c, and d also satisfy ad – bc $\neq$ 0 then S(z) is called a Mobius transformation.