A review for the second exam

Theorem, important examples and definitions to review

1. The definition of distribution and tempered distribution
2. The definition of derivative of a distribution, Fourier transform, product of a distribution and a function, convolution of a distribution and a function, support of a distribution, fundamental solution of a differential equation, with appropriate examples
3. The definition of convergence in distribution sense
4. The distributional derivative of order m of a function f which is differentiable m times coincides with the distribution $u_{f^{(m)}}$
5. The product of g and u_f is u_{fg}
6. The convolution of g and u_f is u_{f*g}
7. $f * \delta_0 = f$
8. The FT of δ_0 is $\equiv 1$
9. The solution in $\mathcal{S}'(\mathbf{R}^n)$ of $\Delta u = 0$ are polynomials
10. The sequence $k^n e^{-\pi k^2 |x|^2}$ converge to δ_0 in distribution sense
11. The real and imaginary parts of a holomorphic function are harmonic

Problems

1. Let $f(x) = \dfrac{1 - \cos x}{x}$; evaluate $f * f * f * f$.
2. Evaluate the derivative, (in distribution sense) of $[x]$ (the integer part of x). Evaluate its Fourier transform too.

3. Evaluate the derivative (in distribution sense) of $f(x) = \sum_{k=0}^{\infty} \frac{e^{kx}}{k}$. Evaluate (if possible) its Fourier transform too

4. Find the derivative, in distribution sense, of the function $f(x) = \begin{cases} e^{x} & \text{if } x < a, \\ e^{-x} & \text{if } x \geq a \end{cases}$.

5. Let $f_k(x) = \sin(kx)$. Find the limit, in distribution sense, of u_{f_k}.

6. Show that the function $u(x, y) = e^{x} \cos y$ is harmonic. This is not a polynomial, but this does not contradict the Theorem proved in class because ...

7. Prove or disprove: the solutions in $\mathcal{S}'(\mathbf{R}^n)$ of the equation $\Delta u = p(x)$, where $p(x)$ is a polynomial, are polynomials.

8. Verify that $u = \displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k} \delta_k$ is a tempered distribution. What is its support? Then evaluate the convolution of u and $f(x) = \sin x$.

9. Find the support of the distributions a) $u(\psi) = \displaystyle\int_{-\pi}^{\pi} |x| \psi^{(4)}(x) dx$ and b) $u(\psi) = \psi(0) + \psi'(1) + |x|$

10. Evaluate the Fourier transform of the fundamental solution of $P(D)u - 2\Delta u + u$, where $P(D)u = \displaystyle\sum_{j=1}^{n} (\partial_j^4 u + 2\partial_j^2 u + u)$

11. a) Let $u = \mathrm{Re} f$ and $v = \mathrm{Im}\ f$, where $f(x, y) = f(x + iy)$ is holomorphic in $\mathbf{C}$. Recalling that $\partial_x f + i\partial_y f = 0$, verify that a) uv is harmonic. and b) $\nabla u . \nabla v = 0$.

    c) Use the previous problem to show that $e^{2(x^2 - y^2)} \cos(xy) \sin(xy)$ is harmonic.

12. Find a and b for which $P(x, y) = ax(x^2 - y^2) + by(xy)$ is harmonic