VECTOR SPACE $\mathbb{R}^n$ – IMPORTANT CONCEPTS (review)

① Subspace of $\mathbb{R}^n$: a set U of vectors in $\mathbb{R}^n$ such that :

(a) U is closed under addition ($x, y \in U \Rightarrow x+y \in U$)

(b) U is closed under scalar multiplication ($x \in U, a \in \mathbb{R} \Rightarrow ax \in U$)

② Linear combination of vectors $x_1, x_2, \dots, x_k$: a vector of the form $t_1x_1 + t_2x_2 + \dots + t_kx_k$, for any scalars $t_1, t_2, \dots, t_k$

③ Span of vectors $x_1, x_2, \dots, x_k \in \mathbb{R}^n$: the set of all linear combinations of $x_1, x_2, \dots, x_k$; that is,

$$\operatorname{span}\{x_1, x_2, \dots, x_k\} = \{t_1x_1 + t_2x_2 + \dots + t_kx_k \,:\, t_1, \dots, t_k \in \mathbb{R}\}$$

④ Linear independence : a set $\{x_1, x_2, \dots, x_k\} \subseteq \mathbb{R}^n$ is linearly independent if

$$t_1x_1 + \dots + t_kx_k = 0 \quad \text{implies} \quad t_1 = \dots = t_k = 0 \,;$$

i.e. the zero vector can be written as a lin. combination of $x_1, x_2, \dots, x_k$ in one and only one way.

⑤ Basis for a subspace U : a set of vectors $\{x_1, \dots, x_k\}$ that is

(a) linearly independent, and

(b) spans U , i.e. $U = \operatorname{span}\{x_1, x_2, \dots, x_k\}$.

⑥ Dimension of a subspace U : the number of vectors in any basis for U.

Orthogonal sets and expansion in $\mathbb{R}^n$

Orthogonal vectors X, Y: $\boxed{X \cdot Y = 0}$

orthogonal set $\{X_1, X_2, \dots, X_k\}$:

(1) $X_i \cdot X_j = 0$ for all $i \neq j$

(2) $X_i \neq 0$ for all i

orthonormal set $\{X_1, \dots, X_k\}$:

(1) $\{X_1, \dots, X_k\}$ is an orthogonal set

(2) $\|X_i\| = 1$ for all i

Properties of orthogonal sets

Let $\{F_1, F_2, \dots, F_m\}$ be an orthogonal set. Then:

① Pythagora's theorem

$$\boxed{\|F_1 + \dots + F_m\|^2 = \|F_1\|^2 + \dots + \|F_m\|^2}$$

② $\{F_1, \dots, F_m\}$ is an independent set.

③ If $X \in \operatorname{span}\{F_1, \dots, F_m\}$, then

$$\boxed{X = \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \dots + \frac{X \cdot F_m}{\|F_m\|^2} F_m}$$

Fourier expansion