Homework 1- part II. Due Wed, Sept. 22

1. How would the construction of the Lebesgue measure in $\mathbf{R}^2$ change if we assume at the beginning that the measure of the unit square $[0,1] \times [0,1]$ is =2? What would the measure of a circle of radius 1 be? Why?

2. Prove that the set $S = [0,1] \times \{0\}$ is Lebesgue measurable in $\mathbf{R}^2$ and $m^*(S) = 0$

3. Recall that the **support** of a function $f : \mathbf{R}^n \to \mathbf{R}$ is the closure of the set $\{x \in \mathbf{R}^n : f(x) = 0\}$. Prove that if $f : \mathbf{R}^n \to \mathbf{R}^n$ has support in a set of Lebesgue measure =0, then $\int_{\mathbf{R}^n} f(x)dx = 0$.

   (this problem is not easy, but you can try first and then look for the proof e.g. in the book pages)

4. a) Evaluate $\displaystyle\int_{\mathbf{R}^2} xy\chi_{\{[0,1]\times[0,1]\}} dm^*(x,y)$, (or if you prefer, $\int_{[0,1]\times[0,1]} xy dm^*(x,y)$).

   b) Evaluate $\displaystyle\int_{\mathbf{R}^2} xy\chi_{\{[0,1]\times[0,1]-\mathbf{Q^2}\}} dm^*(x,y)$.

5. Prove that the function $f(x) = \dfrac{\sin x}{x}$ is in $L^p(\mathbf{R})$ for every $1 < p \leq \infty$. Use the fact that $|f(x)| \leq ...$ when $|x| \leq \frac{\pi}{2}$ and $|f(x)| \leq ...$ when $|x| > \frac{\pi}{2}$.