Solutions to the problems from Lecture 2.

1. The key to understand this problem lies in the following fact:

> If a sequence of numbers x_n is decreasing, then either the sequence decreases all the way to $-\infty$, or it has a finite limit.

In other words, if $x_1 > x_2 > x_3 > \ldots$ etc, *and* it is bounded from below, then it must have a finite limit. This fact we accept intuitively; for more details we'd need a course in Analysis.

So for this problem what we outlined in parts b), c), and d), are the following steps: in b) we show that the only *possible* limit is $\sqrt{2}$ (but there may not be a limit at all). In c) we show that x_1, x_2, etc, are all bounded from below by the value $\sqrt{2}$. In d) we show that the sequence is decreasing. From the reasoning outlined in the paragraph above, the sequence *will* have a limit, and that limit will necessarily be equal to $\sqrt{2}$.

Proof of b). Since we assume $x_n \to L$, then $x_{n+1} \to L$ also. Substituting L for both values in the recursive formula and then solving for L we find that $L^2 = 2$. This has two possible solutions, namely $L = \sqrt{2}$ and $L = -\sqrt{2}$. This second case is impossible since our seed is $x_0 = 1$, which is positive, and the recursive formula will only give us positive values.

Proof of c). Work *backwards* from what you want to prove until you arrive at a true formula, *taking care that all steps are reversible!!!!* If the steps are not reversible, then this process does not work.

$$\sqrt{2} \leq \frac{1}{2}\left(x + \frac{2}{x}\right) \tag{1}$$

$$2\sqrt{2} \leq x + \frac{2}{x} \tag{2}$$

$$2\sqrt{2}\,x \leq x^2 + 2 \tag{3}$$

$$0 \leq x^2 - 2\sqrt{2}\,x + 2 \tag{4}$$

$$0 \leq (x - \sqrt{2})^2 \tag{5}$$

The last is a true statement (a square can't ever be negative), and all steps are reversible. The problem step is between the second and third equations, because we multiplied by x. Since x is positive that did not change the direction of the inequality, and the steps are reversible.

Proof of d). Like in c), work backwards from what you want to prove until you arrive at $\sqrt{2} < x$, taking care that all steps are reversible.

Here is a different, more elegant proof for parts c) and d). For part c), let $f(x) = \frac{1}{2}(x + 2/x)$. We want to show $f(x) > \sqrt{2}$ if $x > 0$. Taking derivatives we get $f'(x) = \frac{1}{2}(1 - 2/x^2)$, and $f''(x) = 2/x^3$. Since the second derivative is always positive for $x > 0$, the graph is concave up. The first derivative is zero when $x = \sqrt{2}$, and consequently that value is a global minimum. This means that $f(x) \geq f(\sqrt{2}) = \sqrt{2}$.

For d) we set $g(x) = x - f(x)$. We want to show that $g(x) > 0$ if $x > \sqrt{2}$. We find that $g(\sqrt{2}) = 0$, and that $g'(x) = \frac{1}{2} + 2/x^2$, which is always positive. Therefore $g(x)$ is strictly increasing, and so $g(x) > 0$ if $x > \sqrt{2}$.

2. The vector whose entries are all equal to 1 is what we have in mind here. Since the rows of M add up to 1, then Mv will be equal to v again.

Here is the reason for this exercise. The important matrices are the stochastic matrices. They have *columns* which add up to 1, not rows. For stochastic matrices we would like to answer the following question: do they have an equilibrium? Namely, if A is a stochastic matrix, can we find a vector v such that $Av = v$? This question seems difficult to solve, but here is a fact from Linear Algebra to help us:

> If a matrix A has an equilibrium ($Av = v$), then the transpose matrix A^t also has an equilibrium ($A^t w = w$), and vice-versa.

(In Linear Algebra language: A and A^t have the same eigenvalues.)

So instead of showing that a stochastic matrix A has an equilibrium we instead show that $A^t = M$ has an equilibrium. This is much simpler, because the rows of M add up to 1, and the vector all formed by 1's is the equilibrium configuration.

3. a) Here are the equations (remember that the first and last nodes have fixed values 5 and 3, respectively).

$$\begin{aligned} a_2(n+1) &= \frac{a_3(n)}{2} + \frac{5}{2} \\ a_3(n+1) &= \frac{a_2(n)}{2} + \frac{a_4(n)}{2} \\ a_4(n+1) &= \frac{a_3(n)}{2} + \frac{3}{2} \end{aligned}$$

b) The matrix can be read from the above equations, by reading the coefficients of the various terms (the coefficient is zero if the term is not present). So

$$M = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}.$$

c) Like in question 1b), we use the recursive formulas, replacing $a_2(n)$ by a_2, $a_3(n)$ by a_3, etc. We obtain a system with three equations and three unknowns, and solving the system will give us $a_2 = 9/2$, $a_3 = 4$, $a_4 = 7/2$.

4. a) The equations for $a_2(n+1)$, $a_3(n+1)$, and $a_4(n+1)$ are like before, except that the 5 is now replaced by $a_1(n)$, and the 3 is replaced by $a_5(n)$. The new equations are

$$a_1(n+1) = \frac{a_1(n)}{2} + \frac{a_2(n)}{2}$$
$$a_5(n+1) = \frac{a_4(n)}{2} + \frac{a_5(n)}{2}.$$

b) Stack the equations on top of each other, with the $(n+1)$ guys on the left, and the (n) guys on the right, then add them all up. Because on the right you get half-values, but each appearing twice, the end result is

$$a_1(n+1) + \cdots + a_5(n+1) = a_1(n) + \cdots + a_5(n).$$

By induction, this means that the sum of all the nodes is always the same. Since at time zero they sum to 17, that is the answer.

c) Reading from the equations:

$$M = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Please note that this is a stochastic matrix.

d) Use the same trick of substituting $a_1(n)$ by its limit a_1, etc, in the recursive system. That will give us five equations and five unknowns. It turns out that one of the equations is redundant (can be obtained from the others), so you need an extra equation. The extra equation is given by $a_1 + a_2 + \ldots + a_5 = 17$, and if you are careful in solving this system you will find your answer to be 17/5 for all five nodes.

5. The graph looks like a ladder resting against a wall at forty-five degrees. The expression for the function is

$$V(S) = \max\{0, E - S\}.$$

This is because the put option only has any value to you if the stock price S is *lower* than the strike price E, in which case the option is worth the difference, $E - S$. Otherwise the option is worthless.

6. The graph is the sum of the graphs for a call and a put, and has equation $V(S) = |S - E|$, the absolute value of $S - E$.