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Lecture 5. Differential equations and one-period, two-asset markets.

Consider the following problem: You deposit S dollars in the bank, with interest rate r (in percent per year) compounded continuously. How much money will you have after a time of one year has elapsed?

Here's one way to approach this problem. Let $y = f(t)$ be the quantity of money at time t (measured in years), where we conveniently assume that $t = 0$ is the moment we first deposited the money. Then of course $f(0) = S$, and we are looking for $f(1)$.

Choose n a (large) natural number, and break the time interval $[0, 1]$ into n equal pieces, each of size $1/n$. Instead of compounding interest continuously we will compound it only at the times k/n. Let S_k be the money accumulated up to time k/n. We know that $S_0 = S$, and if the number n is large enough, then $S_k \approx f(k/n)$. S_k can be computed quite easily using the recursive formula:

$$S_{k+1} = S_k \left(1 + \frac{r}{n}\right), \qquad S_0 = S.$$

If we substitute the many formulas into the next ones, we find that

$$f(1) \approx S_n = S \left(1 + \frac{r}{n}\right)^n.$$

This approximation becomes an equality when we let n tend to infinity (a nice limit to compute), and we obtain

$$f(1) = S\, e^r.$$

We can easily scale the time: if we want to know $f(T)$, the interest is scaled to rT, and the answer becomes $f(T) = S\, e^{rT}$.

I want now a different way to compute this answer. Fix a time t, and a small time displacement Δt. At time t there are $f(t)$ dollars in the bank. At time $t + \Delta t$ we approximate the total money in the bank by

$$f(t + \Delta t) \approx f(t)\,(1 + r\Delta t).$$

Notice: the interest rate is r per unit of time, and since only a time of Δt has elapsed, we obtain the last formula. Regrouping terms we get

$$\Delta f = f(t + \Delta t) - f(t) \approx f(t)\, r\, \Delta t.$$

As we make the elapsed time become instantaneous, the approximation becomes an equality, Δt becomes dt, and Δf becomes df. Overall, the equation becomes

$$df = r\, f\, dt.$$

This is a differential equation relating the instantaneous variation of f given a instantaneous variation of t. The question now becomes: can we find a function f which solves the above equation? For instance, $g(t) = t^2$ **does not** solve the problem, since $dg = 2t\, dt = 2\sqrt{g}\, dt$. On the other hand, $g(t) = C\, e^{rt}$ solves the equation, since $dg = Cr\, e^{rt} dt = r\, g\, dt$. Can we find this solution starting from the equation?

Starting from $df = rf\,dt$, rewrite it as

$$\frac{df}{f} = r\,dt$$

and integrate on both sides:

$$\int \frac{df}{f} = \int r\,dt \Longrightarrow \ln(f) = rt + C_1 \Longrightarrow f = C\,e^{rt}.$$

Here $C = e^{C_1}$. Therefore $f(t) = C\,e^{rt}$, and since $f(0) = S$, we find $C = S$.

A *differential equation* is an expression relating an unknown function f to some of its own derivatives (and possibly some other functions). A *solution* to the equation is a specific function, which can be correctly substituted in the formula.

Example: $(df/dt) = rf$ has $f(t) = e^{rt}$ as a solution. $(d^2f/dt^2) = -f$ has $f(t) = \sin t$ as a solution.

The *order* of the equation is the highest derivative showing up in it. The order of the first example is one, of the second is two.

First and second order equations are extremely important in practice, because they appear in all sorts of examples from "real life" applications. The interest rate example above is one such application. In this course we will concentrate our attention on first order equations.

A first order differential equation is an expression like

$$f'(t) = \frac{df}{dt} = \text{expression involving } f \text{ and possibly other functions.}$$

For instance,

$$\frac{df}{dt} = f^2, \qquad \frac{df}{dt} = \frac{1}{t^2 - f}, \qquad \frac{df}{dt} = -2f, \qquad \frac{df}{dt} = t^2\,f,$$

are all first order equations. We call them *separable* when we can put all symbols containing f to the left, and all symbols containing t to the right. Of the examples above only the second is not separable. Separable equations are easy to solve, and they are always solved in the same way: separate the variables and integrate on both sides.

$$\frac{df}{dt} = t^2\,f \Longrightarrow \frac{df}{f} = t^2\,dt \Longrightarrow \int \frac{df}{f} = \int t^2\,dt.$$

Integrating we obtain $\ln(f) = t^3/3 + C_1$, and now we exponentiate both sides to get $f(t) = C\,e^{t^3/3}$.

The constants appearing are always found with help from the *initial data*: a known value of the function f at a known time t (usually $t = 0$). In the interest rate example we had $f(0) = S$, and we determined the value of the constant C from the initial data knowledge. A differential equation plus initial data constitute an *initial value problem* (IVP).

Financial interlude. Let's talk again about one period, two-asset markets. Again, one of our securities is a bond (interest rate r), and the other is a generic risky asset (say, a stock). If at expiry date T the bond pays 1, then it must have cost e^{-rT}. *We will assume now there are only two possible future states of the world.* The stock price is S_0, and future payoffs are S_1 and S_2. We assume $S_1 < S_2$. The matrix for this market is

$$\overline{D} = \begin{pmatrix} -e^{-rT} & 1 & 1 \\ -S_0 & S_1 & S_2 \end{pmatrix}$$

The price vector must be obtainable from the payout matrix D when applied to a state-price vector ψ. We need to find the vector ψ.

$$\begin{pmatrix} e^{-rT} \\ S_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ S_1 & S_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

In other words

$$\begin{aligned} e^{-rT} &= \psi_1 + \psi_2 \\ S_0 &= S_1\,\psi_1 + S_2\,\psi_2. \end{aligned}$$

Here we know the values S_0, S_1, S_2, and e^{-rT}, and we want to find ψ_1 and ψ_2. Solving the system we find that

$$\begin{aligned} \psi_1 &= \frac{e^{-rT}\,S_2 - S_0}{S_2 - S_1} \\ \psi_2 &= \frac{S_0 - S_1\,e^{-rT}}{S_2 - S_1} \end{aligned}$$

Because there's no arbitrage the two values ψ_1, ψ_2 must be positive. Since the denominators are positive, the numerators must be positive. That happens exactly when

$$S_1 < e^{rT}\,S_0 < S_2.$$

This last equation is saying that if we collect the money S_0 and instead of buying the stock we instead buy the bond, then the bond payoff is somewhere in between the payoffs given by the stock.

Back to math. Differential equations are an example of a deterministic system. If we interpret (as we are doing) t as time, and f as a quantity changing in time, then the equation describes a law governing the speed of change (df/dt) of the quantity f. If we know the speed of change, and we know the initial quantity $f(0)$, then we must know all future states of f, and so f is completely determined. These observations are encoded in the following theorem.

Theorem. *Given a function of two variables $G(t, z)$ which is continuous and has a continuous derivative with respect to z, then the IVP*

$$\frac{df}{dt} = G(t, f(t)), \qquad f(0) = f_0$$

has only one solution.

We don't need all the hypotheses as they appear in the theorem, but there's enough generality in the theorem to cover most of the interesting cases.

Problems.

1. Solve each of the differential equations below with the same initial data $f(0) = 1$.
a) $df/dt = 5f - 3$; b) $df = 2t\, f^2\, dt$; c) $df = e^f\, dt$; d) $df = (1-f)\, dt$.

2. Verify that the function $f(t) = t^{3/2}$, $t \geq 0$ solves the equation

$$\frac{df}{dt} = \frac{3}{2} f^{1/3} \qquad f(0) = 0.$$

Find a second solution for the IVP, and explain why the theorem does not apply to this case.

3. Assuming $r = .06$ per year and $S = S_0 = 100$, compute the total in the bank after one year if a) you compound interest continuously; b) you compound interest daily.

4. In the finance part, show that the given values for ψ_1 and ψ_2 indeed solve the system. (In other words, solve the system.)

5. In the finance part, show that if the values ψ_1 and ψ_2 are not positive, then there's an arbitrage opportunity in the market. (Hint: if $S_0\, e^{rT} \leq S_1$, describe an arbitrage portfolio. Likewise, if $S_2 \leq S_0\, e^{rT}$.)